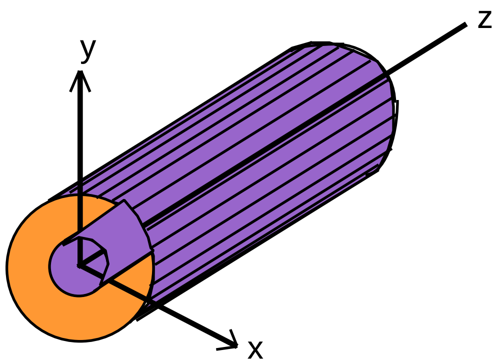
EE-611 Microwave Communications
Supplementary Notes
TEM modes in Coaxial Lines



Coaxial Line

Inner radius = a, Outer radius =b

Transverse Laplacien in cylindrical coordinates

$$\nabla_t^2 \Phi(\rho,\varphi) = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi(\rho,\varphi)}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi(\rho,\varphi)}{\partial\varphi^2} = 0$$

Because of symmetry, no φ variation

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi(\rho,\varphi)}{\partial\rho}\right) = 0$$

$$\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi(\rho,\varphi)}{\partial\rho}\right) = 0$$

Integrating once and arranging the resulting equation

$$\rho\frac{\partial\Phi(\rho,\varphi)}{\partial\rho} = C_1$$

$$\partial\Phi(\rho,\varphi) = C_1\frac{\partial\rho}{\rho}$$

integrating second time

$$\int\partial\Phi(\rho,\varphi) = C_1\int\frac{\partial\rho}{\rho} = C_1\ln(\rho) + C_2$$

$$\Phi(\rho) = C_1\ln(\rho) + C_2$$

Using boundary conditions to evaluate integration constants

at $\rho=a$, $\Phi(a) = V_o$

$$V_o = C_1\ln(a) + C_2$$

at $\rho=b, \quad \Phi(b)=0$

$$0 = C_1 \ln(b) + C_2$$

$$C_2 = -C_1 \ln(b)$$

$$V_o = C_1 \ln(a) - C_1 \ln(b)$$

$$V_o = C_1 \ln\left(\frac{a}{b}\right)$$

$$C_1 = \frac{V_o}{\ln\left(\frac{a}{b}\right)}$$

$$C_2 = -\frac{V_o}{\ln\left(\frac{a}{b}\right)} \ln(b)$$

Finally,

$$\Phi(\rho) = \frac{V_o}{\ln\left(\frac{a}{b}\right)} \ln(\rho) - \frac{V_o}{\ln\left(\frac{a}{b}\right)} \ln(b)$$

$$\Phi(\rho) = \frac{V_o}{\ln\left(\frac{a}{b}\right)} \ln\left(\frac{\rho}{b}\right)$$

The Electric field is

$$\mathbf{E}(\rho,z) = \mathbf{e}(\rho)\, e^{-j\beta z} = - \quad \Phi(\rho)\, e^{-j\beta z}$$

$$= -\left(\frac{\partial}{\partial\rho}\Phi(\rho)\right)\mathbf{a}_\rho\, e^{-j\beta z}$$

$$= -\frac{V_o}{\ln\left(\frac{a}{b}\right)} \frac{1}{\rho}\, e^{-j\beta z}\, \mathbf{a}_\rho$$

$$= \frac{V_o}{\ln\left(\frac{b}{a}\right)} \frac{1}{\rho}\, e^{-j\beta z}\, \mathbf{a}_\rho$$

Corresponding Magnetic Field is

$$\mathbf{H}(\rho,z) = Y_o\, \mathbf{a}_z \times \mathbf{e}\, e^{-j\beta z}$$

$$\mathbf{H}(\rho,z) = Y_o\, \mathbf{a}_z \times \left(\frac{V_o}{\ln\left(\frac{b}{a}\right)} \frac{1}{\rho}\, e^{-j\beta z}\, \mathbf{a}_\rho\right)$$

$$\mathbf{H}(\rho,z) = Y_o \frac{V_o}{\ln\left(\frac{b}{a}\right)} \frac{1}{\rho}\, e^{-j\beta z}\, \mathbf{a}_\varphi$$

We can define a traveling voltage wave from

$$V = -\int \mathbf{E} \cdot d\mathbf{l}$$

$$V_a - V_b = V = -\int_b^a \left(\frac{V_o}{\ln\left(\frac{b}{a}\right)} \frac{1}{\rho} e^{-j\beta z} \mathbf{a}_\rho \right) \cdot (d\rho \mathbf{a}_\rho)$$

$$V = \frac{V_o}{\ln\left(\frac{b}{a}\right)} \ln(\rho) \Big|_a^b e^{-j\beta z} = V_o e^{-j\beta z}$$

Similarly, a traveling current waveform can be defined

$$I = \int_S \mathbf{J} \cdot dS = \int_C (\mathbf{n} \times \mathbf{H}_s) \cdot d\mathbf{l}$$

$$I = \int_0^{2\pi} \left(\mathbf{a}_\rho \times \left\{ Y_o \frac{V_o}{\ln\left(\frac{b}{a}\right)} \frac{1}{\rho} e^{-j\beta z} \mathbf{a}_\varphi \right\} \right)_{\rho=a} \cdot (a d\phi \mathbf{a}_z)$$

$$I = Y_0 \frac{V_o}{\ln\left(\frac{b}{a}\right)} \frac{1}{a} 2\pi a\, e^{-j\beta z} = 2\pi Y_o \frac{V_o}{\ln\left(\frac{b}{a}\right)} e^{-j\beta z} \quad \text{(A/m)}$$

Finally, the characteristic impedance is define as

$$Z_c = \frac{1}{2\pi} Z_o \ln\left(\frac{b}{a}\right)$$

$$Z_c = \frac{1}{2\pi} \eta_o \ln\left(\frac{b}{a}\right)$$

$$Z_c \approx \frac{60}{\sqrt{\varepsilon_r}} \ln\left(\frac{b}{a}\right)$$