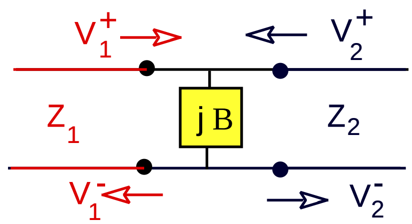
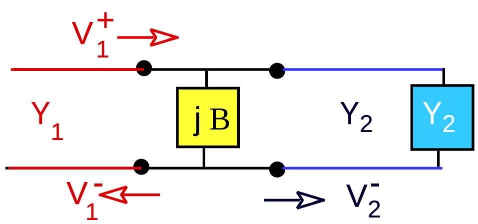
ECECS-611
Microwave Communications

Example: Scattering Matrix
Find the scattering matrix and the generalized scattering matrix for a susceptance connected as a shunt element between two transmission lines of characteristic impedance Z_1 and Z_2.

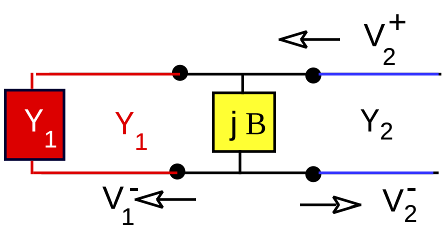


Using unnormalized coordinates, we can write by definition and inspection



$$S_{11} = \left.\frac{V_1^-}{V_1^+}\right|_{V_2^+=0} = \frac{Y_1 - Y_L}{Y_1 + Y_L} = \frac{Y_1 - (Y_2 + jB)}{Y_1 + (Y_2 + jB)}$$

Similarly



$$S_{22} = \left.\frac{V_2^-}{V_2^+}\right|_{V_1^+=0} = \frac{Y_2 - Y_L}{Y_2 + Y_L} = \frac{Y_2 - (Y_1 + jB)}{Y_2 + (Y_1 + jB)}$$

Note that as expected $S_{11} \neq S_{22}$.
Also, by definition

$$S_{21} = \left.\frac{V_2^-}{V_1^+}\right|_{V_2^+=0}$$

Equating the voltages and manipulating terms

$$V_2^- = V_1^+ + V_1^- = V_1^+\left(1 + \frac{V_1^-}{V_1^+}\right) = V_1^+\,(1 + S_{11})$$

Substituting S_{22} from above

$$S_{21} = \left.\frac{V_2^-}{V_1^+}\right|_{V_2^+=0} = \frac{2\,Y_1}{Y_2 + Y_1 + jB}$$

Also

$$S_{12} = \left.\frac{V_1^-}{V_2^+}\right|_{V_1^+ = 0}$$

Equating the voltages and manipulating terms

$$V_1^- = V_2^+ + V_2^- = V_2^+ \left(1 + \frac{V_2^-}{V_2^+} \right) = V_2^+ \,(1 + S_{22})$$

Substituting S_{11} from above

$$S_{12} = \left.\frac{V_1^-}{V_2^+}\right|_{V_1^+=0} = \frac{2\, Y_2}{Y_2 + Y_1 + jB}$$

Note that in this case $S_{12} \neq S_{21}$.

Using normalized voltages, the generalized scattering matrix elements become

$$S'_{11} = \left.\frac{b_1}{a_1}\right|_{a_2=0} = \frac{V_1^- \sqrt{Y_1}}{V_1^+ \sqrt{Y_1}} = \frac{Y_1 - (Y_2 + jB)}{Y_1 + (Y_2 + jB)} = S_{11}$$

$$S'_{22} = \left.\frac{b_2}{a_2}\right|_{a_1=0} = \frac{V_2^- \sqrt{Y_2}}{V_2^+ \sqrt{Y_2}} = \frac{Y_2 - (Y_1 + jB)}{Y_2 + (Y_1 + jB)} = S_{22}$$

and

$$S'_{12} = \left.\frac{b_1}{a_2}\right|_{a_1=0} = \frac{V_1^- \sqrt{Y_1}}{V_2^+ \sqrt{Y_2}} = \sqrt{\frac{Y_1}{Y_2}}\, S_{12} = \sqrt{\frac{Y_1}{Y_2}}\, \frac{2\, Y_2}{Y_2 + Y_1 + jB} = \frac{2 \sqrt{Y_1\, Y_2}}{Y_2 + Y_1 + jB}$$

$$S'_{21} = \left.\frac{b_2}{a_1}\right|_{a_2=0} = \frac{V_2^- \sqrt{Y_2}}{V_1^+ \sqrt{Y_1}} = \sqrt{\frac{Y_2}{Y_1}}\, S_{21} = \sqrt{\frac{Y_2}{Y_1}}\, \frac{2\, Y_1}{Y_2 + Y_1 + jB} = \frac{2 \sqrt{Y_1\, Y_2}}{Y_2 + Y_1 + jB}$$

Note that for this case $S'_{12} = S'_{21}$, [S] matrix is symmetric.

If the two port impedances are identical, both scattering parameters are the same and $S_{12} = S_{21}$.