Microwave Resonant Circuits and Structures.

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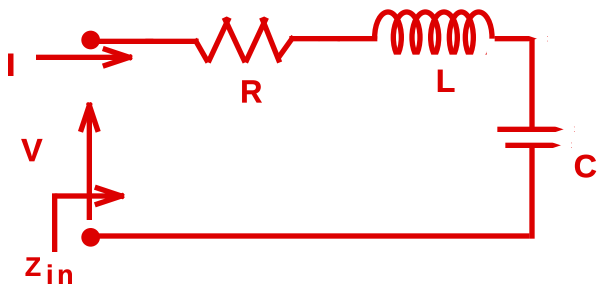
In order to understand the properties of the microwave structures, well known series and parallel RLC circuits will be reviewed and their relation to microwave resonant circuits will be made.

Series RLC circuit



Consider first the series connected RLC circuit as shown in the figure. The input impedance can be written as

$$Z_{in} = R + j\omega L + \frac{1}{j\omega C} \tag{1}$$

This can be written as

$$Z_{in} = R + j\omega L\left(1 - \frac{1}{\omega^2 LC}\right)$$

$$Z_{in} = R + j\omega L\left(1 - \frac{\omega_o^2}{\omega^2}\right) \tag{2}$$

Here ω_o=1/sqrt(LC) is the resonant frequency. Let

$$\omega = \omega_o + \Delta\omega \tag{3}$$

where $\Delta\omega$ is the deviation of the frequency from resonant frequency ω_o. Substituting Eq.3 into Eq.2

$$Z_{in} = R + j(\omega_o + \Delta\omega)L\left(1 - \frac{\omega_o^2}{(\omega_o + \Delta\omega)^2}\right)$$

$$Z_{in} = R + j(\omega_o + \Delta\omega)L\left(1 - \frac{\omega_o^2}{\omega_o^2\left(1 + \frac{\Delta\omega}{\omega_o}\right)^2}\right)$$

$$Z_{in} = R + j(\omega_o + \Delta\omega)L\left(1 - \frac{1}{\left(1 + 2\frac{\Delta\omega}{\omega_o} + \left(\frac{\Delta\omega}{\omega_o}\right)^2\right)}\right)$$

Neglecting second order terms and expanding 1/(1+x) as 1-x for x<<1 ($x=\Delta\omega/\omega_o$)

$$Z_{in} \approx R + j(\omega_o+\Delta\omega)L\left[1-\left(1-\left(2\frac{\Delta\omega}{\omega_o}\right)\right)\right]$$

$$Z_{in} = R + j(\omega_o+\Delta\omega)L\left(2\frac{\Delta\omega}{\omega_o}\right)$$

$$Z_{in} \approx R + j(2\,\Delta\omega L) \qquad (4)$$

The input impedance can be written in terms of power loss and energy stored

$$Z_{in} = \frac{P_l + 2j\omega(W_m - W_e)}{\frac{1}{2}II^*} \qquad (5)$$

Where

$$P_l = \frac{1}{2}RII^* \qquad W_m = \frac{1}{4}LII^* \qquad W_e = \frac{1}{4}\frac{II^*}{\omega^2 C}$$

The quality factor can be written as

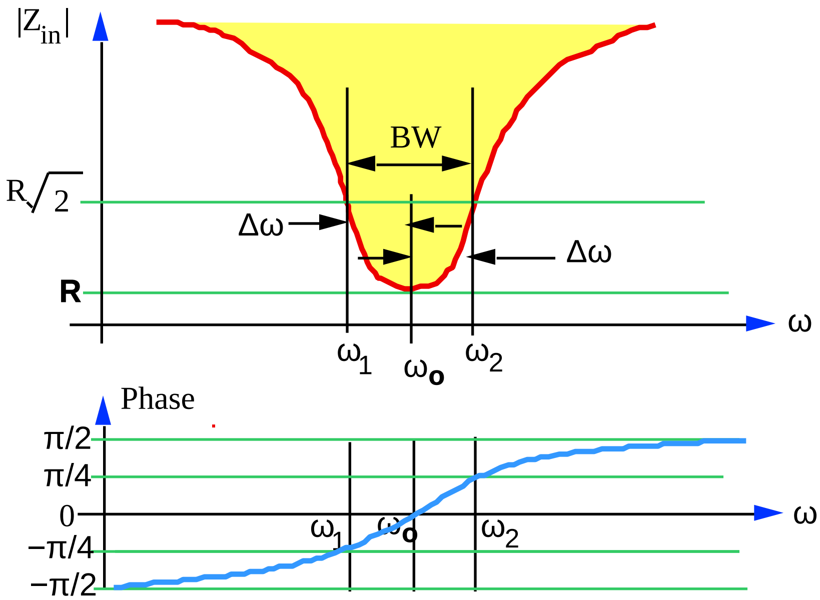
$$Q = \frac{\omega(\text{total energy stored})}{\text{Power loss}} = \omega\left.\frac{\frac{1}{2}\frac{II^*}{\omega^2 C}}{\frac{1}{2}RII^*}\right|_{\omega=\omega_o} = \frac{1}{\omega_o RC}$$

$$Q = \frac{L}{\omega_o RLC} = \frac{L\,\omega_o^2}{\omega_o R} = \frac{\omega_o L}{R}$$

Therefor as the series resistance decreases the quality factor increases.
Eq.4 can now be written in terms of the Q of the circuits.

$$Z_{in} = R\left(1 + j\,2\frac{\Delta\omega}{\omega_o}Q\right) \qquad (6)$$

The magnitude and phase of Eq.6 can now be plotted against frequency ω.



Note that at resonance when $\omega = \omega_o$, the input impedance magnitude is equal to R and the phase is 0^o. Let $\omega_1 = \omega_o - \Delta\omega$ and $\omega_2 = \omega_o + \Delta\omega$ be the frequencies at which real and imaginary parts of Z_{in} are equal, i.e.,

$$\frac{2\Delta\omega}{\omega_o}Q = \frac{2Q}{\omega_o}(\omega_o - \omega_1) = 1$$

$$\omega_1 = \omega_o\left(1 - \frac{1}{2Q}\right)$$

and

$$\frac{2\Delta\omega}{\omega_o}Q = \frac{2Q}{\omega_o}(\omega_2 - \omega_o) = 1$$

$$\omega_2 = \omega_o\left(1 + \frac{1}{2Q}\right)$$

the difference between $\omega_2 - \omega_1$ is defined as the bandwidth of the tuned circuit

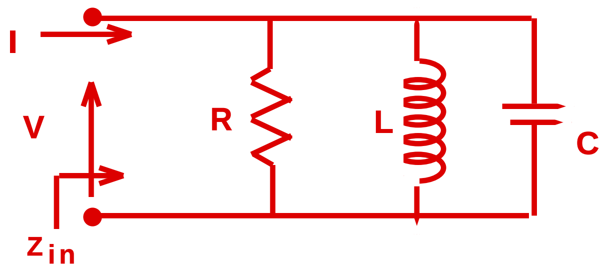
$$\omega_2 - \omega_1 = \omega_o\left(1 + \frac{1}{2Q}\right) - \omega_o\left(1 - \frac{1}{2Q}\right) = \omega_o\frac{1}{Q}$$

and the fractional bandwidth BW is defined as

$$BW = \frac{\omega_2 - \omega_1}{\omega_o} = \frac{1}{Q}$$

Note that the bandwidth is inversely proportional to the Q of the resonant structure. The higher the Q of the circuit, the narrower is the bandwidth of the circuit.

Parallel RLC circuit



The same procedure is used for the derivation of Z_{in}.

$$Z_{in} = \left[\frac{1}{R} + j\omega C - \frac{1}{j\omega L}\right]^{-1}$$

$$Z_{in} = \left[\frac{1}{R} + j\omega C\left(1 - \frac{1}{\omega^2 LC}\right)\right]^{-1}$$

$$Z_{in} = \left[\frac{1}{R} + j\omega C\left(1 - \frac{\omega_o^2}{\omega^2}\right)\right]^{-1}$$

$$Z_{in} = \left[\frac{1}{R} + j(\omega_o + \Delta\omega)\, C\left(1 - \frac{\omega_o^2}{(\omega_o + \Delta\omega)^2}\right)\right]^{-1}$$

$$Z_{in} = \left[\frac{1}{R} + j(\omega_o + \Delta\omega)\, C\left(1 - \frac{\omega_o^2}{\omega_o^2\left(1 + 2\frac{\Delta\omega}{\omega_o} + \left(\frac{\Delta\omega}{\omega_o}\right)^2\right)}\right)\right]^{-1}$$

$$Z_{in} \approx \left[\frac{1}{R} + j(\omega_o + \Delta\omega)\, C\left(1 - \left(1 - 2\frac{\Delta\omega}{\omega_o}\right)\right)\right]^{-1}$$

$$Z_{in} \approx \left[\frac{1}{R} + j(\omega_o)\, C\left(2\frac{\Delta\omega}{\omega_o}\right)\right]^{-1}$$

$$Z_{in} \approx \left[\frac{1}{R} + j\, C\, 2\Delta\omega\right]^{-1}$$

In terms of energies and power loss

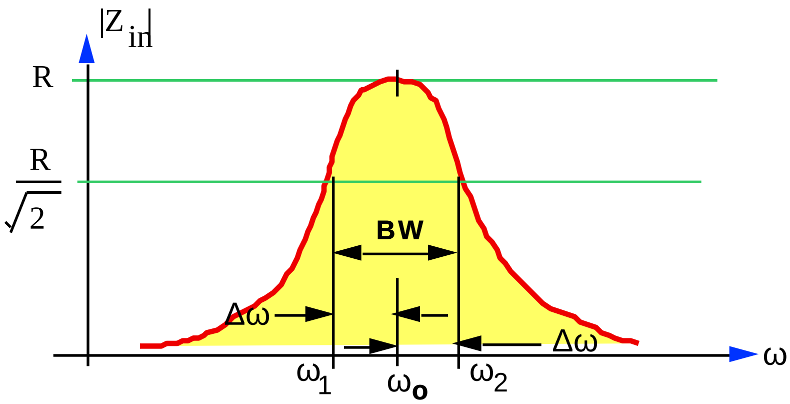
$$P_l = \frac{1}{2}\frac{|V|^2}{R} \qquad W_e = \frac{1}{4}C|V|^2 \qquad W_m = \frac{1}{4}L|I|^2 = \frac{1}{4}L\left|\frac{V}{j\omega L}\right|^2 = \frac{1}{4}\frac{|V|^2}{\omega_o L}$$

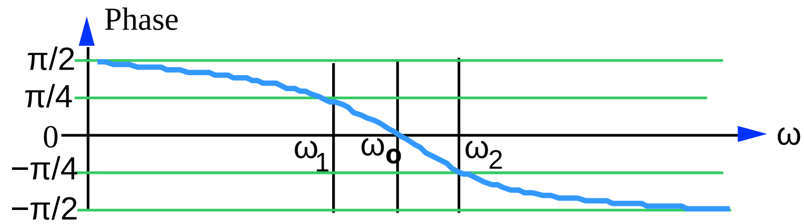
$$Q = \frac{\omega\,(\text{Total energy stored})}{\text{Total power loss}} = \frac{\omega_o\left(\frac{1}{2}C|V|^2\right)}{\frac{1}{2}\frac{|V|^2}{R}} = \omega_o CR = \frac{R}{\omega_o L}$$

In the case of parallel circuit, the Q increases with increasing R. Z_{in} can be written in terms of Q as

$$Z_{in} = \frac{1}{(G + j2\Delta\omega\, C)} = \frac{R}{1 + j\,2\,\Delta\omega RC} = \frac{R}{1 + j\,Q\,\dfrac{\Delta\omega}{\omega_o}}$$

Magnitude and phase of Z_{in} as a function of ω for the parallel RLC is plotted in the figure below.



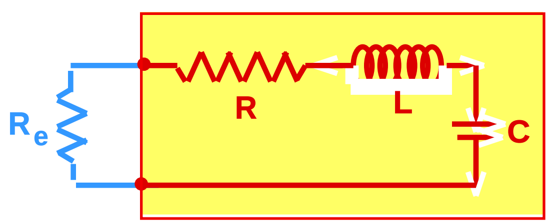


Loaded Q

The circuits given above are isolated circuit elements. In actual circuitry, the resonant structure is connected to the outside world or signal is coupled from an external system. The resonant circuit is usually connected to a transmission line and thus sees at its terminals equivalent resistance R_e.

For the Series circuit, the external resistance is in series with the R of the isolated system. The total resistance is then

R_t=R+R_e



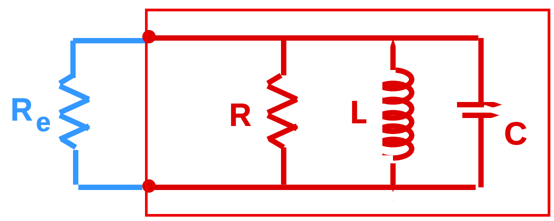
The equivalent Q of the system including the external resistance is now defined as the loaded Q, or Q_L.

$$Q_L = \frac{1}{\omega_o C R_t} = \frac{1}{\omega_o C(R_e+R)} = \frac{1}{\omega_o C R_e + \omega_o C R} = \frac{1}{\frac{1}{Q_e} + \frac{1}{Q_o}}$$

$$\frac{1}{Q_L} = \frac{1}{Q_e} + \frac{1}{Q_o}$$

Here $Q_o=1/\omega CR$ is the unloaded Q and the $Q_e=1/\omega CR_e$ is the external Q of the cavity. The measurement provides the loaded Q_L which is always smaller than the unloaded Q_o of the resonant structure.

For the parallel circuitry, the external resistance is in parallel with the isolated resistance.



The total resistance is $R_t=[RR_e/(R+R_e)]$. The loaded Q for this case is

$$Q_L = \omega_o C\, R_t = \omega_o C \left(\frac{R_e\, R}{R_e + R} \right) = \omega_o C \frac{1}{\left(\frac{1}{R_e} + \frac{1}{R} \right)} = \frac{1}{\left(\frac{1}{\omega_o C R_e} + \frac{1}{\omega_o C R} \right)}$$

$$Q_L = \frac{1}{\left(\frac{1}{Q_e} + \frac{1}{Q_o} \right)}$$

$$\frac{1}{Q_L} = \frac{1}{Q_e} + \frac{1}{Q_o}$$

This is the same result as the Series case.