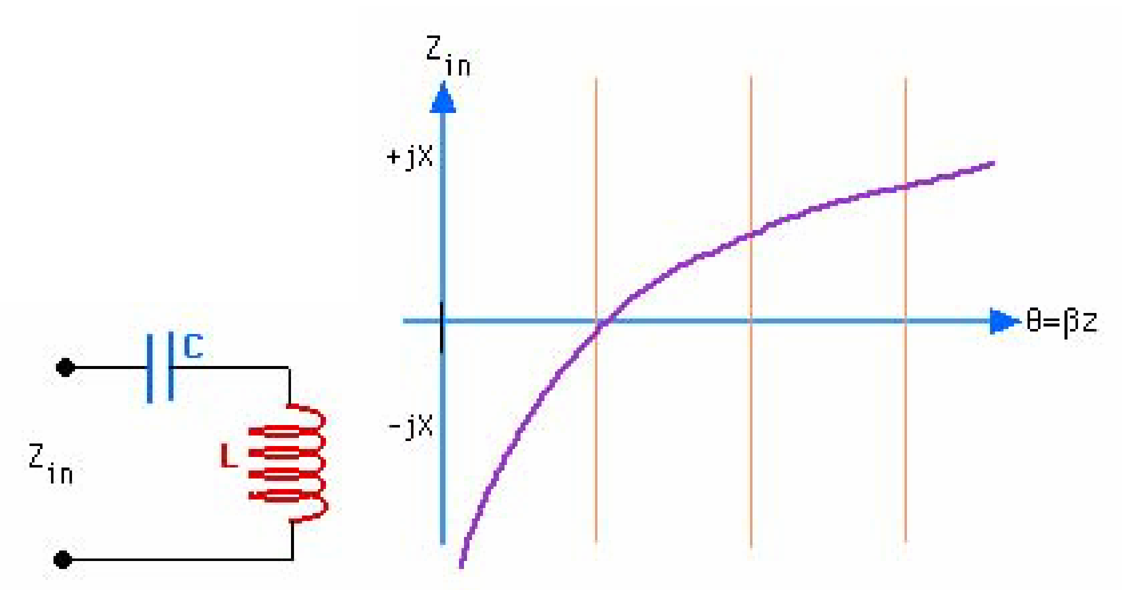
ECE-611
Microwave Communications
Terminated Transmission Lines

Basic difference between lumped elements and distributed circuit elements is the frequency response of a given circuit.

For example, when a series LC network is considered (neglecting the losses), the input impedance is written as

$$Z_{in}=j\omega L+1/j\omega C$$



The frequency response of the input impedance for the series LC network is plotted in the figure above. Here the impedance is imaginary all the time, it is infinity at $\omega=0$, is capacitive up to a frequency $\omega=1/(LC)^{1/2}$ and becomes positive (inductive) above that is frequency and approaches $j\infty$ as the frequency increases. Thus, the impedance goes through zero once. Lumped parameters exhibit a different response as will be shown below.

Transmission Lines

On a transmission line, the total Voltage and Current at z are

$$V(z) = V^{+} e^{-j\beta z} + V^{-} e^{+j\beta z}$$

$$I(z) = I^{+} e^{-j\beta z} + I^{-} e^{+j\beta z}$$

$$I(z) = \frac{1}{Z_c} [V^{+} e^{-j\beta z} - V^{-} e^{+j\beta z}]$$

A load Z_L is connected at the output terminals of the transmission line. By definition, the Load Impedance is

$$Z_L = \frac{V(0)}{I(0)}$$

$$Z_L = \frac{V^+ + V^-}{\frac{1}{Z_c}(V^+ - V^-)} = Z_c \frac{1 + \frac{V^-}{V^+}}{1 - \frac{V^-}{V^+}}$$

Define Reflection coefficient (complex quantity) as

$$\Gamma_L = \frac{V^-(0)}{V^+(0)} = \frac{V^-}{V^+} = \rho_L \, e^{j\theta}$$

$$Z_L = \ = Z_c \frac{1 + \Gamma_L}{1 - \Gamma_L}$$

Solving for Γ_L, we obtain

$$\Gamma_L = \frac{Z_L - Z_c}{Z_L + Z_c}$$

Reflection coefficient at any length z=-d is

$$\Gamma(z = -d) = \frac{V^-(d)}{V^+(d)} = \frac{V^- \, e^{-j\beta d}}{V^+ \, e^{+j\beta d}} = \Gamma_L \, e^{-j2\beta d} = \rho_L \, e^{j\theta} \, e^{-j2\beta d} = \rho_L \, e^{j(\theta - 2\beta d)}$$

Magnitude is determined by the load reflection coefficient and does not change but phase changes

By definition, the input Impedance at z=-d is

$$Z_{in}(z=-d) = \frac{V(z=-d)}{I(z=-d)}$$

$$Z_{in} = \frac{V^+e^{+j\beta d} + V^-e^{-j\beta d}}{\frac{1}{Z_c}(V^+e^{+j\beta d} - V^-e^{-j\beta d})} = Z_c \frac{1 + \frac{V^-}{V^+} \, e^{-j2\beta d}}{1 - \frac{V^-}{V^+} \, e^{-j2\beta d}} = Z_c \frac{1 + \Gamma_L \, e^{-j2\beta d}}{1 - \Gamma_L \, e^{-j2\beta d}}$$

Manipulating the equation, Z_{in} is simplified to

$$Z_{in} = Z_c \frac{Z_L + j Z_c \tan(\beta d)}{Z_c + j Z_L \tan(\beta d)}$$

In terms of Admittance parameters,

$$Y_{in} = Y_c \frac{Y_L + j Y_c \tan(\beta d)}{Y_c + j Y_L \tan(\beta d)}$$

Various load conditions

$Z_L = Z_c$ (matched load)

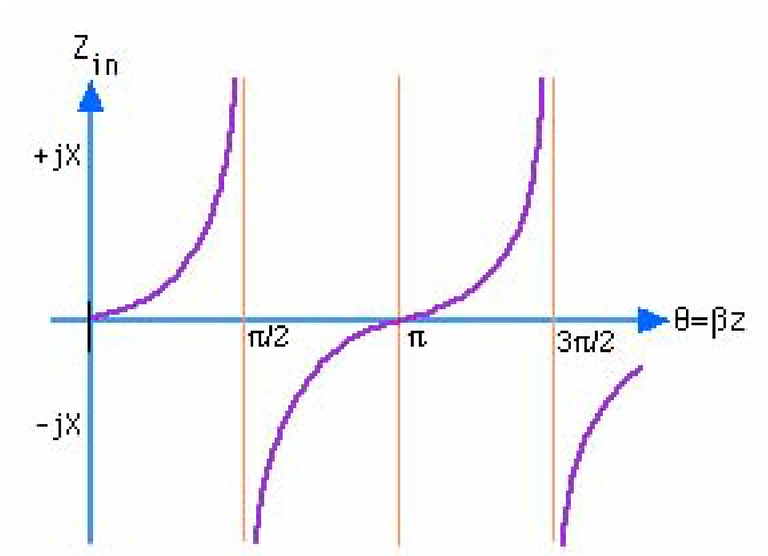
$$\Gamma_L = \frac{Z_c - Z_c}{Z_c + Z_c} = 0, \quad \Gamma(z = -d) = \Gamma_L e^{-j2\beta d} = 0$$

$$Z_{in} = Z_c \frac{Z_c + j Z_c \tan(\beta d)}{Z_c + j Z_c \tan(\beta d)} = Z_c$$

$Z_L = 0 \quad \Rightarrow \quad$ short circuit

$$\Gamma_L = \frac{0 - Z_c}{0 + Z_c} = -1$$

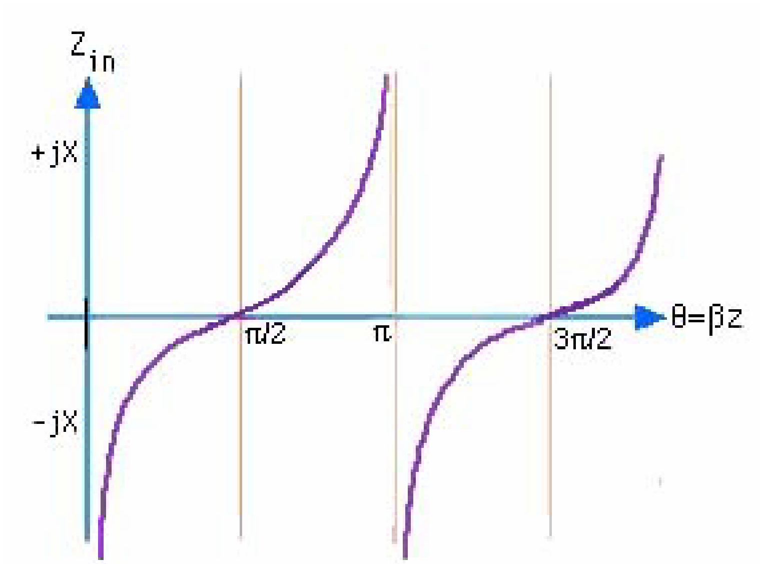
$$Z_{in} = Z_c \frac{0 + j Z_c \tan(\beta d)}{Z_c + j\, 0 \tan(\beta d)} = j Z_c \tan(\beta d)$$



$Z_L = \infty \quad \Rightarrow \quad$ open circuit

$$\Gamma_L = \frac{Z_L - Z_c}{Z_L + Z_c}\Bigg|_{Z_L \to \infty} = \frac{Z_L}{Z_L}\Bigg|_{Z_L \to \infty} = +1$$

$$Z_{in} = Z_c \frac{Z_L + j Z_c \tan(\beta d)}{Z_c + j Z_L \tan(\beta d)}\Bigg|_{Z_L \to \infty} = Z_c \frac{Z_L}{j Z_L \tan(\beta d)}\Bigg|_{Z_L \to \infty} = -j Z_c \cot(\beta d)$$



$$d = n\frac{\lambda}{2} \quad \to \beta d = \frac{2\pi}{\lambda} n\frac{\lambda}{2} = n\pi \quad \text{(multiples of half–wavelength)}$$

$$\Gamma(-d) = \Gamma_L e^{-j2\beta d} = \Gamma_L e^{-j2n\pi} = \Gamma_L$$

$$Z_{in} = Z_c \frac{Z_L + j Z_c \tan(\pi)}{Z_c + j Z_L \tan(\pi)} = Z_c \frac{Z_L}{Z_c} = Z_L \quad [\text{since } \tan(\pi) = 0]$$

The impedance repeats itself every $\lambda/2$.

$$d = \frac{\lambda}{4} \quad \to \beta d = \frac{2\pi}{\lambda} \frac{\lambda}{4} = \frac{\pi}{2} \quad \text{(quarter–wavelength)}$$

$$\Gamma(-d) = \Gamma_L e^{-j2\beta d} = \Gamma_L e^{-j\pi} = -\Gamma_L$$

$$Z_{in} = Z_c \frac{Z_L + j Z_c \tan\left(\frac{\pi}{2}\right)}{Z_c + j Z_L \tan\left(\frac{\pi}{2}\right)} = Z_c \frac{Z_L + j Z_c \infty}{Z_c + j Z_L \infty} = Z_c \frac{Z_c}{Z_L} = \frac{Z_c^2}{Z_L} \quad \left[\text{since } \tan\left(\frac{\pi}{2}\right) = \infty\right]$$

The impedance is inverted. Short circuit becomes an open circuit and an open circuit becomes a short circuit if the line length is quarter wavelength.

As can be seen from these results that the impedance of a transmission line with an arbitrary load repeats itself every $\lambda/2$.