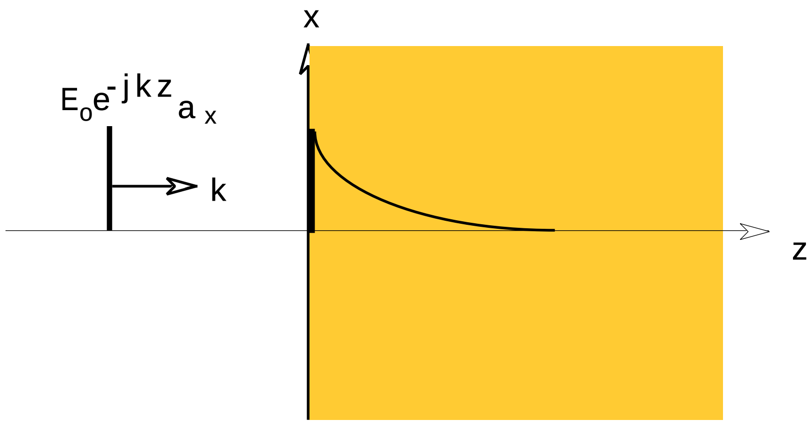
EE-611
Supplementary Notes
Infinity Conductivity Approximation

Assume a plane wave is incident on a metal with a conductivity σ. The metal occupies the space $0<z<\infty$ as shown in Figure below.



The fields in the metal can be written as

$$\mathbf{E}_m = T\, E_o\, e^{-\gamma z}\, \mathbf{a}_x \tag{4.1a}$$

$$\mathbf{H}_m = \frac{T\, E_o}{Z_m}\, e^{-\gamma z}\, \mathbf{a}_y \tag{4.1b}$$

Here T is the transmission coefficient. The incident fields are

$$\mathbf{E}_{in} = E_o\, e^{-j k_o z}\mathbf{a}_x + \Gamma\, E_o\, e^{+j k_o z}\, \mathbf{a}_x \tag{4.2a}$$

$$\mathbf{H}_{in} = \frac{E_o}{\eta_o}\, e^{-j k_o z}\, \mathbf{a}_y - \frac{\Gamma\, E_o}{\eta_o}\, e^{+j k_o z}\, \mathbf{a}_y \tag{4.2b}$$

Using the continuity of **E** and **H** fields,

$$\Gamma = \frac{Z_m - \eta_o}{Z_{m.} + \eta_o} \tag{4.3a}$$

$$T = \frac{2\, Z_m}{Z_m + \eta_o} \tag{4.3b}$$

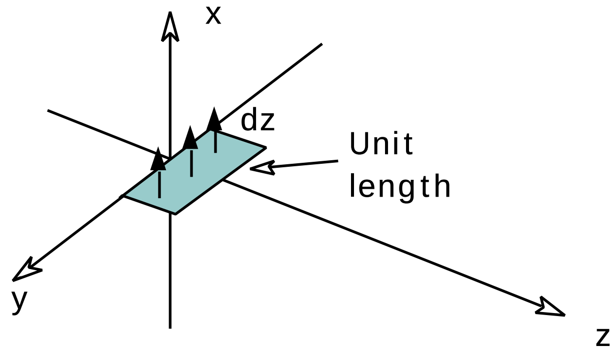
For a good conductor, $\eta_o >> Z_m$. Therefore $\Gamma \approx -1$ and $T \approx 0$. For a perfect conductor s=• and Γ=1 and T =0 and the total magnetic field at the surface of the metal is

$$\mathbf{H}_t = 2E_o\, \mathbf{a}_y \tag{4.4}$$

Now the current density in the conductor is (from Ohm's law)

$$\mathbf{J} = \sigma\,\mathbf{E} = \sigma\,T\,E_o\,e^{-\gamma z}\,\mathbf{a}_x \qquad (4.5)$$

The total linear current density per unit width (unit width is in y direction as shown in figure below) in the metal is



$$\mathbf{J}_s = \int_0^\infty \mathbf{J}\,dz = \sigma\,T\,E_o\,\mathbf{a}_x \int_0^\infty e^{-\gamma z}\,dz = \frac{\sigma\,T\,E_o\,\mathbf{a}_x}{\gamma} \qquad (4.6)$$

Substituting Z_m and γ,

$$\mathbf{J}_s = \frac{2\,\sigma\,Z_m^2\,E_1}{(Z_m + \eta_o)^2 (j\,\omega\,\mu)}\;\mathbf{a}_x \qquad (4.7)$$

As $\sigma \to \infty$, $Z_m \to 0$ and $\sigma Z_m^2 \to j\omega\mu$, the limiting $\mathbf{J}_s$ becomes

$$\mathbf{J}_s = \frac{2\,E_o}{\eta_o}\;\mathbf{a}_x \qquad (4.8)$$

If we were to calculate the total current density on the this conductor at z=0, we will obtain

$$\mathbf{J}_s = \mathbf{n} \times \mathbf{H} = (-\,\mathbf{a}_z) \times \frac{2\,E_1}{\eta_o}\;\mathbf{a}_y = \frac{2\,E_1}{\eta_o}\;\mathbf{a}_x \qquad (4.9)$$

Both calculations give the same result.

For finite conductivity, average power transmitted into the conductor per unit area is given by

$$P_t = \frac{1}{2}\,\mathrm{Re}\,(\,\mathbf{E} \times \mathbf{H}^*)\cdot \mathbf{a}_z\,\big|_{z=0}$$

$$P_t = \frac{1}{2}\,\mathrm{Re}\left\{\,T E_1 \mathbf{a}_x \times \left(\frac{T E_1}{Z_m}\,\mathbf{a}_y\right)^*\right\}\cdot \mathbf{a}_z$$

$$P_t = \frac{1}{2}\,T\,T^*\;E_1\,E_1^*\;\mathrm{Re}\left\{\frac{1}{Z_m}\right\}$$

$$P_t = \frac{1}{2}\,T\,T^*\;E_1\,E_1^*\,(\sigma\delta_s) \qquad (4.10)$$

Using the expression

$$\sigma T T^* = \frac{4\,\sigma\, Z_m Z_m^*}{(Z_m + \eta_o)(Z_m + \eta_o)^*} \approx \frac{4\,\sigma\, |Z_m|^2}{\eta_o^2} = \frac{4\,\sigma\, 2\left(\frac{1}{\sigma\delta_s}\right)^2}{\eta_o^2}$$

$$\sigma T T^* = \frac{8}{\sigma\delta_s^2\, \eta_o^2} \tag{4.11}$$

Equation 4.10 can be written as

$$P_t = \frac{1}{2}\,\frac{8}{\sigma\delta_s^2\,\eta_o^2}\, E_1\, E_1^*\, \delta_s = \frac{1}{2}\left(\frac{2\,E_1}{\eta_o}\right)\left(\frac{2\,E_1}{\eta_o}\right)^*\frac{1}{\sigma\delta_s}$$

$$P_t = \frac{1}{2}\,\mathrm{Re}\{H_t H_t^* Z_m\} = \frac{1}{2}\,\mathrm{Re}\{\, \mathbf{J}_s \cdot \mathbf{J}_s^*\, Z_m\,\} \tag{4.12}$$

Above equation 4.12 would have been obtained if the following average ohmic loss per unit length is calculated:

$$P_t = \frac{1}{2}\int_0^\infty \mathbf{E}\cdot\mathbf{J}^*\, dz = \frac{1}{2\sigma}\int_0^\infty \mathbf{J}\cdot\mathbf{J}^*\, dz = \frac{1}{2\sigma}\int_0^\infty \mathbf{H}_t\cdot\mathbf{H}_t^*\, dz \tag{4.13}$$

Here $\mathbf{H}_t$ is the tangential magnetic field at the surface of the conductor.

Whenever good conductors are involved in a boundary value problem, the following excellent approximations can be made in calculating the losses associated with good conductors :

1. To star with, the conductors can be assumed to have *infinite conductivity*
2. Once the solution of the Maxwell's equations are found subject to the boundary conditions that the metals have infinite conductivity ($\sigma=\infty$), the fields that are found are not any different than fields that would have been obtained from the rigorous solution of the Maxwell's equation in the dielectric and metal regions.
3. In order to find conductor losses, the fields found in (2), are used to find the surface current densities at the conductor boundaries using

$$\mathbf{J_s} = \mathbf{n}\times\mathbf{H}\big|_{\text{surface}} \tag{4.14}$$

4. Using Eq. 4.12, the corresponding conductor losses can then be found.
5. Finally, the attenuation constant is then given by

$$\alpha = \frac{P_l}{2\,P_t} \tag{4.15}$$

where P_t is the total incident power.