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ON THE SOLUTION OF THE GENERAL EQUATION OF THE FOURTH DEGREE.

By A. LODGE, St. John's College, Oxford.

REFERRING to the paper by M. Legoux, which appeared in this Journal in October last, on an application of the special form of determinant, previously discussed by Mr. Glaisher and Mr. Scott in this Journal in 1879 and 1880, I have, by means of this form of determinant, been led to an extremely simple solution of the general equation of the fourth degree, which appears to have been previously overlooked.

Transforming the equation

$$ax^4 + 4bx^3 + 6cx^2 + 4dx + e = 0 \quad\text{............}(1)$$

so as to lose its second term, by putting $y = ax + b$, we obtain

$$y^4 + 6Hy^2 + 4Gy + F = 0 \quad\text{................}(2).$$

Let us now identify this expression with the determinant

$$\begin{vmatrix} y, & p, & q, & r \\ r, & y, & p, & q \\ q, & r, & y, & p \\ p, & q, & r, & y \end{vmatrix}.$$

Splitting this determinant into factors in the manner shown by Mr. Scott, it reduces to

$$\begin{vmatrix} y+q, & p+r \\ p+r, & y+q \end{vmatrix} \times \begin{vmatrix} y-q, & p-r \\ r-p, & y-q \end{vmatrix}$$

$$= \{(y+q)^2 - (p+r)^2\}\{(y-q)^2 + (p-r)^2\}$$

$$= (y^2 + q^2 - 2pr)^2 - (2qy - p^2 - r^2)^2 \quad\text{..........................}(3)$$

$$= y^4 - 2y^2(q^2+2pr) + 4qy(p^2+r^2) + (q^2-2pr)^2 - (p^2+r^2)^2 \ldots(4).$$

Comparing coefficients in (4) and (2), we obtain

$$\left.\begin{aligned} q^2 + 2pr &= -3H \\ q(p^2+r^2) &= \quad G \\ (q^2-2pr)^2 - (p^2+r^2)^2 &= \quad F \end{aligned}\right\} \quad\text{...............}(5),$$

whence $\quad 4q^6 + 12Hq^4 + (9H^2 - F)q^2 - G^2 = 0 \quad\text{........}(6).$

By making use of the values

$$F = a^2 I - 3H^2, \quad G^2 = a^2(HI - aJ) - 4H^3,$$

where I, J are the two invariants of the quartic, this equation may be written in either of the forms

$$4q^6 + 12Hq^4 + (12H^2 - a^2 I)\, q^2 - G^2 = 0 \;\ldots\ldots\ldots (6')$$

or

$$4\left(\frac{q^2 + H}{a}\right)^3 - I\,\frac{q^2 + H}{a} + J = 0 \;\ldots\ldots\ldots\ldots (6'').$$

Again, substituting from (5) in (3), we obtain

$$(y^2 + 3H + 2q^2)^2 - \left(2qy - \frac{G}{q}\right)^2 = 0;$$

therefore

$$\left\{(y+q)^2 + 3H + q^2 - \frac{G}{q}\right\}\left\{(y-q)^2 + 3H + q^2 + \frac{G}{q}\right\} = 0;$$

that is

$$\left\{(ax + b + q)^2 + 3H + q^2 - \frac{G}{q}\right\} \times \left\{(ax + b - q)^2 + 3H + q^2 + \frac{G}{q}\right\} = 0 \quad \ldots (7).$$

On finding any value of q^2 from (6) and substituting it in any of the equations (7), we thus split the biquadratic into two quadratic factors, each of which can be at once solved for the roots.

I have given the above method of arriving at equations (7) and the reducing cubic (6), as it was the way in which I was myself led to them; but the first of equations (7) is so simple, and, when once arrived at, so self-evident, that it might have been sufficient to have written it down immediately after (2).

On expanding it, and subtracting (2) from it, we arrive at once at the reducing cubic (6).

The roots of the biquadratic are most intimately connected with those of the reducing cubic. Of these latter, one is positive, and the other two are either both positive, both negative, or both imaginary. Moreover, in the first case two or all of the roots may be equal, and in the second case the two negative roots may be equal. Also any one or more of the roots may be zero. To each of these cases corresponds some special peculiarity in the roots of the biquadratic.

To investigate the general conditions for real or imaginary roots of the biquadratic, let α, β be the roots corresponding to the first factor of (7), and γ, δ the roots corresponding to the second factor.

Then $$\tfrac{1}{4}a^2(\alpha-\beta)^2 = \frac{G}{q} - (q^2+3H)$$

and $$\tfrac{1}{4}a^2(\gamma-\delta)^2 = -\frac{G}{q} - (q^2+3H).$$

Hence all the roots will be real, or all imaginary, if

$$(q^2+3H)^2 - \frac{G^2}{q^2} \text{ is positive;}$$

and two of the roots will be real and two imaginary, if

$$(q^2+3H)^2 - \frac{G^2}{q^2} \text{ is negative.}$$

Taking the first case, and substituting from equation (6), we have

$$(q^2+3H)^2 > 4q^4 + 12Hq^2 + 12H^2 - a^2I;$$

therefore $$a^2I > 3(q^2+H)^2;$$

therefore $$I > 3\phi^2, \text{ where } a\phi = q^2+H.$$

But (from 6″) $$4\phi^3 - I\phi + J = 0;$$

therefore $$\phi^2(4\phi^2-I)^2 = J^2;$$

therefore $$I^3 - 27J^2 \text{ is positive} \quad\text{(8).}$$

Similarly, in the second case, that is, when there are two real and two imaginary roots,

$$I^3 - 27J^2 \text{ is negative} \quad\text{(9);}$$

and if two of the roots are equal,

$$I^3 - 27J^2 = 0 \quad\text{(10).}$$

(The equal roots must of course be real, unless the quartic is a perfect square, when they can be either real or imaginary.)

But I^3-27J^2 is the discriminant (with its sign changed) of the reducing cubic; therefore, if it is positive, all the roots of the cubic are real; if it is negative, two roots are imaginary; and if it is zero, two roots are equal.

Hence, if all the roots of the reducing cubic are real, the roots of the biquadratic are either all real or all imaginary; if the cubic has two imaginary roots, the biquadratic has two roots real and two imaginary; and if the reducing cubic has two equal roots, so also has the biquadratic.

Further, if all the values of q^2 are not only real, but positive, all the roots of the biquadratic must be real, for all its quadratic factors will have real coefficients, which can only happen when the linear factors involve only real quantities. There will be no negative root of (6) if its coefficients are alternately positive and negative; and therefore the necessary and sufficient conditions for all the roots of the biquadratic to be real are

$$\left.\begin{array}{r} H \text{ negative......(i)} \\ 12H^2 - a^2 I \text{ positive ......(ii)} \\ I^3 - 27J^2 \text{ positive ......(iii)} \end{array}\right\} \text{..............} (11),$$

these conditions simply expressing that the roots of the reducing cubic (6) are all real and positive.

Now the conditions for all the roots of the biquadratic to be real are found, by applying Sturm's theorem, to be

$$\left.\begin{array}{r} H \text{ negative......(i)} \\ 2HI - 3aJ \text{ negative......(ii)} \\ I^3 - 27J^2 \text{ positive ......(iii)} \end{array}\right\} \text{..............} (12);$$

and it is important to reconcile the apparent discrepancy between these two sets of conditions.

We have the identity

$$H(12H^2 - a^2 I) + 3G^2 \equiv a^2 (2HI - 3aJ),$$

since $$G^2 + 4H^3 \equiv a^2 (HI - aJ);$$

therefore {12 (ii)} expresses not only that $12H^2 = a^2 I$ must be positive, but also that it must be greater than $\dfrac{3G^2}{-H}$.

And it is readily proveable that this condition must be fulfilled if the roots of the cubic (6) are all positive; for, let l, m, n be these roots, then

$$l^3 + m^3 + n^3 - 3lmn \equiv (l + m + n)(l^2 + m^2 + n^2 - mn - nl - lm);$$

therefore $$(l + m + n)(mn + nl + lm) - 9lmn$$

$$\equiv (l + m + n)(l^2 + m^2 + n^2) - (l^3 + m^3 + n^3) - 6lmn$$

$$\equiv l(m - n)^2 + m(n - l)^2 + n(l - m)^2,$$

which is positive if l, m, n are all positive; therefore

$$-H(12H^2 - a^2 I) - 3G^2 \text{ must be positive;}$$

that is, $$2HI - 3aJ \text{ must be negative.}$$

But this condition is implicitly contained in those of (11), as can be seen from inspection of the easily verified identity

$$a^2 (I^3 - 27J^2)$$

$$\equiv 12HI(2HI - 3aJ) - 3(2HI - 3aJ)^2 - I^2 (12H^2 - a^2 I);$$

Hence either series of conditions (11) or (12) may be employed at pleasure, but in practice those of (11) would generally, if not always, be most easy of application.

If any of the values of q^2 are negative, there will in general be no real root of the biquadratic, as can at once be seen by writing (7) in the form

$$(y^2 + 3H + 2q^2)^2 - q^2\left(2y - \frac{G}{q^2}\right)^2,$$

which, for a negative value of q^2, is the sum of two squares, and cannot therefore vanish for a real value of y, except in the special case in which both squares vanish simultaneously.

This special case is that of equal roots, for, eliminating q from the two equations

$$2y - \frac{G}{q^2} = 0,$$

$$y^2 + 3H + 2q^2 = 0,$$

we obtain $\qquad y^3 + 3Hy + G = 0.$

But this is the first derived function from (2), and can only vanish with it when (2) has equal roots.

(In this case we know that (6) will also have equal roots, a fact which can also be established by eliminating y from the above simultaneous equations.)

If the cubic has all its roots equal, the biquadratic will have three equal roots. For in this case $I = 0$ and $J = 0$, and (6) reduces to

$$(q^2 + H)^3 = 0.$$

Hence $G^2 + 4H^3 = 0$, and the quartic (7) reduces to

$$\{(ax + b + q)^2 - 4q^2\}(ax + b - q)^2 = 0;$$

that is, $\qquad (ax + b - q)^3(ax + b + 3q) = 0,$

where $\qquad q = +\sqrt{(-H)}$ if G is positive,

and $\qquad q = -\sqrt{(-H)}$ if G is negative.

When one of the roots of the cubic is zero, that is, when $G = 0$, the equation is reducible at once to a quadratic in $(ax + b)^2$. In fact we have

$$(ax + b)^4 + 6H(ax + b)^2 + a^2I - 3H^2 = 0;$$

therefore $\qquad \{(ax + b)^2 + 3H\}^2 - (12H^2 - a^2I) = 0.$

Also, in this case,

$$H(12H^2 - a^2I) = 2HI - 3aJ,$$

and $$a^6(I^3 - 27J^2) = (12H^2 - a^2I)^2(a^2I - 3H^2),$$

since $$4H^3 = a^2(HI - aJ).$$

When two of the roots of the cubic are zero (a particular case of equal roots), the quartic becomes a perfect square; for in this case

$$G = 0, \quad 12H^2 - a^2I = 0,$$

and (6) reduces to $$q^4(q^2 + 3H) = 0,$$

whence (7) becomes $$\{(ax + b)^2 + 3H\}^2 = 0.$$

Lastly, when all the roots of the cubic vanish, the biquadratic reduces to

$$(ax + b)^4 = 0;$$

for in this case all the coefficients H, G, F vanish: a condition which may also be written

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{d} = \frac{d}{e}.$$

A METHOD OF EXPRESSING ANY PARTICULAR ARBITRARY CONSTANT IN THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS IN TERMS OF THE INITIAL CONDITIONS.

Part I.

By E. J. Routh.

THE object of the following paper may be very briefly stated. Given any number of simultaneous differential equations with constant coefficients, it is known that the dependent variables x, y, z, &c. can be expressed in terms of the independent variable t, by means of a series of exponentials real or imaginary. Let one of these exponentials be $x = Pe^{pt}$, then P is a function of the initial values of the variables x, y, &c. and of their differential coefficients. It is here proposed to exhibit this function. Thus, without solving the equations, any one term of the solution can be separated from the others and its value written down, without finding those other terms. The rule is given with some examples in Art. 7, and to this the reader may at once proceed.