Reprinted from the MATHEMATICS MAGAZINE
Vol. 48, No. 4, September, 1975
pp. 211–215

QUARTIC EQUATIONS AND TETRAHEDRAL SYMMETRIES

ROGER CHALKLEY, University of Cincinnati

1. Introduction. In Section 2, we give a short derivation of formulas for the roots of a quartic equation. A closely related representation of the symmetric group S_4 by matrices of size 3×3 is presented in Section 3. Geometric interpretations follow in Section 6.

Throughout, let F be a field in which $1+1\neq 0$. For us, the matrix

$$H=\tfrac{1}{2}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & -1 & -1\\ 1 & -1 & 1 & -1\\ 1 & -1 & -1 & 1\end{bmatrix}$$

is basic. It has $H^2=I$ and $H^{-1}=H$.

Suppose 4×4 matrices B and D over F satisfy $BH = HD$. Then, it is easy to verify: D is a diagonal matrix if and only if B has the form

$$B=\begin{bmatrix}\alpha & \beta & \gamma & \delta\\ \beta & \alpha & \delta & \gamma\\ \gamma & \delta & \alpha & \beta\\ \delta & \gamma & \beta & \alpha\end{bmatrix}. \tag{1}$$

When B is given by (1), the diagonal elements of $D = HBH$ are

$$\begin{aligned}\xi_1&=\alpha+\beta+\gamma+\delta,\\ \xi_2&=\alpha+\beta-\gamma-\delta,\\ \xi_3&=\alpha-\beta+\gamma-\delta,\\ \xi_4&=\alpha-\beta-\gamma+\delta.\end{aligned} \tag{2}$$

2. Formulas for the roots of a quartic equation.

THEOREM. *Suppose a, b, c, r_1, r_2, r_3 are elements of F such that*

$$Y^3+2aY^2+(a^2-4c)Y-b^2=(Y-r_1^2)(Y-r_2^2)(Y-r_3^2) \tag{3}$$

and $r_1r_2r_3=-b$. Then, the quartic equation

$$X^4+aX^2+bX+c=0$$

has four roots ξ_1,ξ_2,ξ_3,ξ_4 in F given by

$$[\xi_1,\xi_2,\xi_3,\xi_4]=[0,r_1,r_2,r_3]H. \tag{4}$$

Proof. Set $\alpha=0$, $\beta=\frac{1}{2}r_1$, $\gamma=\frac{1}{2}r_2$, $\delta=\frac{1}{2}r_3$; define B by (1); and, set

$$X^4+\rho X^2+\sigma X+\tau=\det(XI-B). \tag{5}$$

We expand (5) and use (3) with $r_1r_2r_3=-b$ to obtain

$$\begin{aligned}\rho&=-2(\beta^2+\gamma^2+\delta^2)=(-r_1^2-r_2^2-r_3^2)/2=(2a)/2=a,\\ \sigma&=-8\beta\gamma\delta=-r_1r_2r_3=b,\quad\text{and}\\ \tau&=\beta^4+\gamma^4+\delta^4-2(\beta^2\gamma^2+\beta^2\delta^2+\gamma^2\delta^2)\\ &=(\beta^2+\gamma^2+\delta^2)^2-4(\beta^2\gamma^2+\beta^2\delta^2+\gamma^2\delta^2)\\ &=(r_1^2+r_2^2+r_3^2)^2/16-4(r_1^2r_2^2+r_1^2r_3^2+r_2^2r_3^2)/16\\ &=(-2a)^2/16-4(a^2-4c)/16=c.\end{aligned}$$

For the diagonal elements of $D=HBH$, (2) yields (4). We find

$$X^4+aX^2+bX+c=\det(XI-B)=\det(XI-D)=\prod_{s=1}^{4}(X-\xi_s).$$

This completes the proof.

For other derivations, see [7], [5], [6], and [1]; the notation of [5] and [6] corresponds to a substitution of $-Y$ for Y in (3).

3. A representation for the symmetric group S_4. Let $Q_1, Q_2, \cdots, Q_6$ be the six permutation matrices of size 3×3. Set

$$R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

$$R_3 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \qquad R_4 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

PROPOSITION. *Under multiplication, the twenty-four 3×3 matrices*

(6) $$Q_jR_k, \quad \textit{for} \quad j = 1, \cdots, 6 \quad \textit{and} \quad k = 1, \cdots, 4,$$

form a group which is isomorphic to S_4.

Proof. We define matrices P_j and T_k of size 4×4 by

$$P_j = \begin{bmatrix} 1 & 0 \quad 0 & 0 \\ 0 & & \\ 0 & Q_j & \\ 0 & & \end{bmatrix} \quad \text{and} \quad T_k = \begin{bmatrix} 1 & 0 \quad 0 & 0 \\ 0 & & \\ 0 & R_k & \\ 0 & & \end{bmatrix},$$

for $j = 1, \cdots, 6$ and $k = 1, \cdots, 4$. We shall show that the matrices

(7) $$P_jT_k, \quad \text{for} \quad j = 1, \cdots, 6 \quad \text{and} \quad k = 1, \cdots, 4,$$

form a group isomorphic to S_4; then, the same is clearly true for (6).

Let P be a 4×4 permutation matrix. Then, $2PH$ is obtained from $2H$ by a permutation of the rows of $2H$. Thus, the number of components equal to -1 in the first row of $2PH$ is either zero or two; and, each component of the first column of $2PH$ equals 1. We select k from 1, 2, 3, 4 such that each component of the first row of $2PHT_k$ equals 1. Then, each component of the first column of $2PHT_k$ also equals 1; the four columns of $2PHT_k$ are distinct; and, each of the last three columns of $2PHT_k$ has two components equal to 1 and two components equal to -1. Hence, $2H$ can be obtained from $2PHT_k$ by a permutation of the last three columns of $2PHT_k$. We select j from $1, 2, \cdots, 6$ so that

$$(2PHT_k)P_j^{-1} = 2H \quad \text{and} \quad HPH = P_jT_k^{-1} = P_jT_k.$$

Consequently, HPH is one of the matrices in (7).

Under multiplication, the twenty-four permutation matrices of size 4×4 form a group which is isomorphic to S_4. As P ranges over the permutation matrices of size 4×4, the corresponding matrices HPH form a group isomorphic to S_4; but, these are the matrices of (7). This completes the proof.

4. Permutations of the roots. From Section 2, we have

$$[0, r_1, r_2, r_3] = [\xi_1, \xi_2, \xi_3, \xi_4]H.$$

Thus, we obtain $r_1 - r_2 = \xi_2 - \xi_3$, $r_1 + r_2 = \xi_1 - \xi_4, \cdots$, and

$$(r_1^2 - r_2^2)(r_1^2 - r_3^2)(r_2^2 - r_3^2) = \prod_{1 \leq j < k \leq 4} (\xi_j - \xi_k).$$

(For the discriminant of $X^4 + aX^2 + bX + c$, see [7], pp. 173–174.)

As Q_jR_k ranges over the 3×3 matrices of (6), the formula

$$[r_1', r_2', r_3'] = [r_1, r_2, r_3](Q_jR_k)$$

specifies the elements r_1', r_2', r_3' of F which can be substituted for r_1, r_2, r_3 to satisfy the hypothesis of the Theorem in Section 2. The corresponding roots $\xi_1', \xi_2', \xi_3', \xi_4'$ are given by

$$\begin{aligned} [\xi_1', \xi_2', \xi_3', \xi_4'] &= [0, r_1', r_2', r_3']H = [0, r_1, r_2, r_3](P_jT_k)H \\ &= [0, r_1, r_2, r_3]H(HP_jT_kH) = [\xi_1, \xi_2, \xi_3, \xi_4]P, \end{aligned} \tag{8}$$

where P is the 4×4 permutation matrix $P = HP_jT_kH$.

5. Several observations. Here, let F be the field of complex numbers. Then, each finite group has a character table over F ([4], p. 306). The $n \times n$ matrix M_n of [2] specifies a character table for a cyclic group of order n ([4], ex. 10 of pp. 309, 342); and, for $n = 4$, the formula

$$[\xi_1', \xi_2', \xi_3', \xi_4'] = [0, u_0', v_0', w_0']M_4$$

relates the roots of $X^4 + aX^2 + bX + c$ to elements u_0', v_0', w_0' described in [2]. With (8), we have

$$[0, u_0', v_0', w_0'] = [0, r_1, r_2, r_3](HPH)(HM_4^{-1}).$$

In contrast, the matrix $2H$ specifies a character table for the four-group ([4], ex. 11 of pp. 309, 342). Thus, two nonisomorphic groups of order 4 account for the distinct solution procedures (based on M_4 and H) for a quartic equation.

6. Tetrahedral symmetries. Now, let F be the field of real numbers. Relative to a rectangular cartesian coordinate system, the points $(1, 1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, 1)$ are the vertices of a regular tetrahedron. The matrices of (6) represent isometries which map this tetrahedron onto itself. Each of R_2, R_3, R_4 specifies a half-turn about a line through the midpoints of a pair of opposite edges of the tetrahedron. The isometries represented by $Q_1, Q_2, \cdots, Q_6$ form a group of three rotations and three reflections; they leave the vertex $(1, 1, 1)$ fixed and permute the other three vertices. Thus, the full group of isometries for a regular tetrahedron is conveniently represented by (6); it is isomorphic to S_4. The twelve matrices of (6) with determinant equal to 1 represent all the rotations for the tetrahedron; they correspond to the even permutations in S_4. Thus, the group of direct isometries for a regular tetrahedron is isomorphic to the alternating subgroup of S_4. For other viewpoints, see [3].

References

1. R. Chalkley, Cardan's formulas and biquadratic equations, this MAGAZINE, 47 (1974) 8–14.
2. ———, Circulant matrices and algebraic equations, this MAGAZINE, 48 (1975) 73–80.
3. H. S. M. Coxeter, Introduction to Geometry, Wiley, New York, 1961, pp. 271–272.
4. E. D. Nering, Linear Algebra and Matrix Theory, Wiley, New York, 1970, pp. 292–309.
5. B. L. van der Waerden, Modern Algebra, vol. 1, Ungar, New York, 1953, pp. 180–182.
6. S. Warner, Modern Algebra, vol. 2, Prentice-Hall, Englewood Cliffs, New Jersey, 1965, pp. 574–578.
7. H. Weber, Lehrbuch der Algebra, vol. 1, Chelsea, New York, 1963, pp. 135–136.