CARDAN'S FORMULAS AND BIQUADRATIC EQUATIONS

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1. Introduction. In 1956, Dr. Chao-Hui Yang showed me an interesting way to use a 3×3 cyclic matrix to recall Cardan's formulas for the roots of a cubic equation. Recently, by using a 4×4 cyclic matrix in a similar manner, I discovered analogous formulas for the roots of a biquadratic equation. All the details for a cubic are given in Section 2. My results for a biquadratic are presented in Theorem 1 of Section 4; they are related to other solution techniques in Sections 5 and 6.

As coefficient domain, *suppose F is a field of characteristic* $\neq 2, 3$ *with the property: for each element γ in F, $X^2 = \gamma$ has a solution in F and $X^3 = \gamma$ has a solution in F.* In particular, the quadratic formula is applicable, and each second-degree polynomial over F has a root in F; thus, F contains a principal cube root ω

of unity and a principal fourth root i of unity. For definiteness, F can be the field C of complex numbers.

2. Cardan's formulas. Starting with the cyclic matrix

$$\begin{bmatrix} X & Y & Z \\ Z & X & Y \\ Y & Z & X \end{bmatrix}, \tag{1}$$

we can easily remember to write the identity

$$\begin{aligned} X^3 + (-3YZ)X + (Y^3 + Z^3) \\ = (X + Y + Z)(X + \omega Y + \omega^2 Z)(X + \omega^2 Y + \omega Z). \end{aligned} \tag{2}$$

The left member of (2) equals the determinant D of (1). The right member of (2) follows via row operations on (1). Thus, by addition to the first row (X, Y, Z) of the second row (Z, X, Y) and the third row (Y, Z, X), we see $X + Y + Z$ is a factor of D. With $\omega^3 = 1$, addition to (X, Y, Z) of $\omega^2(Z, X, Y)$ plus $\omega(Y, Z, X)$ shows $X + \omega Y + \omega^2 Z$ is a factor of D; etc..

By replacing Y by $-Y$ and Z by $-Z$ in (2), we obtain

$$X^3 + (-3YZ)X + (-Y^3 - Z^3) = \prod_{s=0}^{2} (X - \omega^s Y - \omega^{2s} Z). \tag{3}$$

Suppose α and β are elements of F. To solve the cubic equation

$$X^3 + \alpha X + \beta = 0, \tag{4}$$

we seek a solution (y_0, z_0) of

$$-3YZ = \alpha \text{ and } -Y^3 - Z^3 = \beta. \tag{5}$$

There are two cases.

(i) Suppose $\alpha \neq 0$ or $\beta \neq 0$. Let t_0 be a nonzero solution in F of

$$T^2 + \beta T + \left(-\frac{\alpha}{3}\right)^3 = 0,$$

and let y_0 be a solution in F of $Y^3 = t_0$. With $y_0 \neq 0$, set $z_0 = -\alpha/3y_0$. Then, (y_0, z_0) is a solution of

$$YZ = -\frac{\alpha}{3} \text{ and } (Y^3)^2 + \left(-\frac{\alpha}{3}\right)^3 = -\beta Y^3.$$

Thus, with $y_0 \neq 0$, we see (y_0, z_0) is a solution of (5).

(ii) Suppose $\alpha = 0$ and $\beta = 0$. Then, we set $y_0 = 0$ and $z_0 = 0$.

We substitute y_0 for Y and z_0 for Z in (3) to obtain

$$X^3 + \alpha X + \beta = \prod_{s=0}^{2} (X - \omega^s y_0 - \omega^{2s} z_0).$$

Consequently, *equation* (4) *has three roots* x_1,x_2,x_3 *in* F *given by*

$$x_{s+1} = \omega^s y_0 + \omega^{2s} z_0, \text{ for } s = 0,1,2.$$

Given a cubic equation $\bar{X}^3 + \alpha_1\bar{X}^2 + \alpha_2\bar{X} + \alpha_3 = 0$ over F, the substitution $\bar{X} = X - (\alpha_1/3)$ reduces it to the form (4). Thus, *any cubic equation over* F *is solvable in* F.

3. The determinant of a cyclic matrix. We shall use the following modification of topics in [2].

LEMMA. *Suppose* K *is a field which contains a primitive nth root* ρ *of unity; let* $K[X_1,\cdots,X_n]$ *be a polynomial ring over* K *in* n *variables; and set*

$$A = \begin{bmatrix} X_1 & X_2 & \cdots & X_{n-1} & X_n \\ X_n & X_1 & \cdots & X_{n-2} & X_{n-1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ X_3 & X_4 & \cdots & X_1 & X_2 \\ X_2 & X_3 & \cdots & X_n & X_1 \end{bmatrix}.$$

Then, $\det A = f_0 f_1 \cdots f_{n-1}$, where

$$f_s = X_1 + \rho^s X_2 + \rho^{2s} X_3 + \cdots + \rho^{(n-1)s} X_n, \text{ for } s = 0,1,\cdots,n-1.$$

Proof. For $k = 1,2,\cdots,n$, let R_k denote the kth row of A. Set

$$R = R_1 + \sum_{k=2}^{n} \rho^{(n-k+1)s} R_k.$$

We find $R = (f_s, \rho^{(n-1)s} f_s, \rho^{(n-2)s} f_s, \cdots, \rho^s f_s)$. Let B denote the matrix obtained when the first row of A is replaced by R. We observe $\det B = \det A$ and f_s divides $\det A$ in $K[X_1,\cdots,X_n]$. The polynomial ring is factorial [1], and the elements $f_0, f_1,\cdots,f_{n-1}$ are irreducible. If j and k are integers with $0 \leqq j < k \leqq n-1$, then $\rho^j \neq \rho^k$ and each common divisor of f_j and f_k is a unit. Thus, the product $f_0 f_1 \cdots f_{n-1}$ divides $\det A$. We set

$$\det A = q f_0 f_1 \cdots f_{n-1}.$$

In terms of total degree, we find

$$n = \deg(\det A) = \deg q + \sum_{s=0}^{n-1} \deg f_s = \deg q + n$$

and $\deg q = 0$. Hence, q belongs to K. In $\det A$ and in $f_0 f_1 \cdots f_{n-1}$, the coefficient of X_1^n is 1. This yields $q = 1$ and completes the proof.

4. Formulas for the roots of a biquadratic equation. With $K = F, n = 4$, and $\rho = i$ (where $i^2 = -1$), the Lemma gives

$$\begin{vmatrix} X & U & V & W \\ W & X & U & V \\ V & W & X & U \\ U & V & W & X \end{vmatrix} = \prod_{s=0}^{3} (X + i^s U + i^{2s} V + i^{3s} W).$$

We expand the determinant and replace U, V, W by $-U, -V, -W$ to obtain

$$\begin{aligned} X^4 &+ (-2V^2 - 4UW)X^2 + (-4U^2V - 4VW^2)X \\ &+ (-U^4 + V^4 - W^4 + 2U^2W^2 - 4UV^2W) \\ &= \prod_{s=0}^{3} (X - i^s U - i^{2s} V - i^{3s} W). \end{aligned} \tag{6}$$

This identity leads to the following result.

THEOREM 1. *Suppose a, b, c are elements of F. When $a = b = c = 0$, set $u_0 = v_0 = w_0 = 0$; otherwise, let v_0 be a nonzero solution in F of*

$$(4V^2)^3 + 2a(4V^2)^2 + (a^2 - 4c)(4V^2) - b^2 = 0, \tag{7}$$

and let u_0, w_0 be elements of F which satisfy

$$UW = -\frac{v_0^2}{2} - \frac{a}{4} \text{ and } U^2 + W^2 = -\frac{b}{4v_0}. \tag{8}$$

Then, the biquadratic equation

$$X^4 + aX^2 + bX + c = 0 \tag{9}$$

has four roots x_1, x_2, x_3, x_4 in F given by

$$x_{s+1} = i^s u_0 + i^{2s} v_0 + i^{3s} w_0, \text{ for } s = 0,1,2,3. \tag{10}$$

Proof. First, we verify (u_0, v_0, w_0) is a solution of

$$4UW = -2V^2 - a, \tag{11}$$

$$4V(U^2 + W^2) = -b, \tag{12}$$

and

$$(U^2 + W^2)^2 + 8UWV^2 = \frac{a^2 - 4c}{4}. \tag{13}$$

For $a = b = c = 0$, this is clear. For $v_0 \neq 0$, we rewrite (7) as

(14) $$4^2V^2\left(\frac{a^2-4c}{4}-2(-2V^2-a)V^2\right)=b^2;$$

then, (u_0, v_0, w_0) is a solution of (7), (8), (11), (12), (14), and

$$4^2V^2\left(\frac{a^2-4c}{4}-8UWV^2\right)=4^2V^2(U^2+W^2)^2;$$

hence, with $v_0 \neq 0$, (u_0, v_0, w_0) is also a solution of (13).

We use (11) to eliminate a from (13); thus, (u_0, v_0, w_0) satisfies

(15) $$-2V^2-4UW=a,$$

(16) $$-4U^2V-4VW^2=b,$$

and

(17) $$-U^4+V^4-W^4+2U^2W^2-4UV^2W=c.$$

We substitute (u_0, v_0, w_0) in (6), (15), (16), and (17) to obtain

$$X^4+aX^2+bX+c=\prod_{s=0}^{3}(X-i^s u_0-i^{2s}v_0-i^{3s}w_0).$$

Consequently, (9) has four roots in F given by (10).

5. Further information. In Theorem 1, the element $4v_0^2$ is a root of

(18) $$Y^3+2aY^2+(a^2-4c)Y-b^2=0.$$

We proceed to relate all three roots of (18) to Theorem 1.

PROPOSITION. *Suppose u_0, v_0, w_0 in F satisfy* (11), (12), *and* (13). *Set*

(19) $$r_1=(1+i)u_0+(1-i)w_0,\ r_2=2v_0,\ r_3=(1-i)u_0+(1+i)w_0.$$

Then, $r_1r_2r_3=-b$ and (18) *has three roots in F given by r_1^2, r_2^2, and r_3^2.*

Proof. We use (19), (11), (13), and (12) to obtain

$$\begin{aligned}(Y-r_1^2)(Y-r_2^2)(Y-r_3^2)&=(Y-r_2^2)(Y^2-8u_0w_0Y+4(u_0^2+w_0^2)^2)\\&=Y^3+(-4v_0^2-8u_0w_0)Y^2+(4(u_0^2+w_0^2)^2+32u_0w_0v_0^2)Y\\&\quad-16v_0^2(u_0^2+w_0^2)^2\\&=Y^3+2aY^2+(a^2-4c)Y-b^2\end{aligned}$$

and

$$r_1r_2r_3=4v_0(u_0^2+w_0^2)=-b.$$

In [4], equation (18) was given as a cubic resolvent for equation (9), and a different solution procedure was established. Next, we derive the solution formulas of [4] from Theorem 1.

THEOREM 2. *Suppose elements* r_1, r_2, r_3 *of* F *satisfy*

$$(Y - r_1^2)(Y - r_2^2)(Y - r_3^2) = Y^3 + 2aY^2 + (a^2 - 4c)Y - b^2 \tag{20}$$

and

$$r_1 r_2 r_3 = -b. \tag{21}$$

Then, equation (9) *has four roots* x_1, x_2, x_3, x_4 *in* F *given by*

$$\begin{aligned} x_1 &= (+r_1 + r_2 + r_3)/2, \\ x_2 &= (+r_1 - r_2 - r_3)/2, \\ x_3 &= (-r_1 + r_2 - r_3)/2, \\ x_4 &= (-r_1 - r_2 + r_3)/2. \end{aligned} \tag{22}$$

Proof. We define u_0, v_0, w_0 in F through

$$4u_0 = (1 - i)r_1 + (1 + i)r_3,\ 2v_0 = r_2,\ 4w_0 = (1 + i)r_1 + (1 - i)r_3. \tag{23}$$

Using (23), (20), and (21), we find

$$\begin{aligned} 4u_0 w_0 + 2v_0^2 &= \frac{r_1^2 + r_2^2 + r_3^2}{2} = -a, \\ 4v_0(u_0^2 + w_0^2) &= r_1 r_2 r_3 = -b, \text{ and} \\ (u_0^2 + w_0^2)^2 + 8u_0 w_0 v_0^2 &= \frac{r_1^2 r_2^2 + r_1^2 r_3^2 + r_2^2 r_3^2}{4} = \frac{a^2 - 4c}{4}. \end{aligned}$$

Thus, (u_0, v_0, w_0) is a solution of (11), (12), and (13). From the proof of Theorem 1, the roots of (9) are given by (10). For $s = 0, 1, 2, 3$, we use (10) and (23) to obtain (22).

6. Several observations. Given λ, μ in F, we specify a solution of

$$UW = \lambda \text{ and } U^2 + W^2 = \mu.$$

For $\lambda = \mu = 0$, set $u_0 = w_0 = 0$; otherwise, let t_0 be a nonzero solution in F of $T^2 - \mu T + \lambda^2 = 0$, let u_0 satisfy $U^2 = t_0$, and set $w_0 = \lambda / u_0$. In this way, (8) can be satisfied.

When $b = 0$ in equation (9), the conditions

$$V = 0,\ UW = -\frac{a}{4}, \text{ and } (U^2 + W^2)^2 = \frac{a^2 - 4c}{4}$$

also specify a solution of (11), (12), and (13) as well as a solution procedure (10) for (9). We can take $\lambda = -a/4$ and μ so $\mu^2 = (a^2 - 4c)/4$. Of course, with $b = 0$, equation (9) can be solved directly as a quadratic in X^2.

Let S_1 be the set of arrangements (first, second, third, fourth) for the four roots of (9); let S_2 be the set of solutions of (11), (12), and (13); and, let S_3 be the set of triples (r_1, r_2, r_3) which satisfy (20) and (21). *There exists a bijection of S_2 onto S_1.* Namely, the mapping from S_2 to S_1 given by (10) is clearly injective. To prove it surjective, suppose (x_1, x_2, x_3, x_4) is an element of S_1; then, $x_1 + x_2 + x_3 + x_4 = 0$; by solving linear equations, we find unique elements u_0, v_0, w_0 in F which satisfy (10); by (6), (u_0, v_0, w_0) is a solution of (15), (16), and (17); hence, (u_0, v_0, w_0) is an element of S_2; etc. Similarly, *there exists a bijection of S_3 onto S_1.* Directly from (19) and (23), *the sets S_2 and S_3 are in one-to-one correspondence.*

Another procedure to deduce a cubic resolvent for (9) is given in [**3**]. Based upon Galois theory, the solution formulas of [**3**] are analogous to (18), (20), (21), and (22). The differences of notation are due to the changes necessitated when Y is replaced in (18) by $-Y$.

Given a biquadratic equation $\bar{X}^4 + a_1\bar{X}^3 + a_2\bar{X}^2 + a_3\bar{X} + a_4 = 0$ over F, the substitution $\bar{X} = X - (a_1/4)$ reduces it to the form (9). Thus, any biquadratic equation over F is solvable in F.

7. An example. In the equation

$$X^4 - 2\beta^2X^2 - 4\alpha^2\beta X + (\beta^4 - \alpha^4) = 0, \tag{24}$$

we suppose α and β are nonzero elements of F. With

$$a = -2\beta^2,\ b = -4\alpha^2\beta,\ c = \beta^4 - \alpha^4, \text{ and } (a^2 - 4c)(-2a) - b^2 = 0,$$

the corresponding cubic resolvent (18) has $-2a$ as a root.

To solve (24) by Theorem 1, we take

$$4v_0^2 = -2a = 4\beta^2,\ v_0 = \beta,\ UW = 0,\ U^2 + W^2 = \alpha^2,\ u_0 = \alpha, \text{ and } w_0 = 0.$$

Thus, by (10), the roots of (24) are

$$x_{s+1} = i^s\alpha + i^{2s}\beta, \text{ for } s = 0, 1, 2, 3. \tag{25}$$

In this situation, Theorem 2 is less direct. First, all three roots

$$2i\alpha^2,\ 4\beta^2,\ -2i\alpha^2$$

of (18) are required. Then, elements r_1, r_2, r_3 of F are needed to satisfy

$$r_1^2 = 2i\alpha^2,\ r_2^2 = 4\beta^2,\ r_3^2 = -2i\alpha^2, \text{ and } r_1r_2r_3 = 4\alpha^2\beta;$$

one choice is $r_1 = (1 + i)\alpha$, $r_2 = 2\beta$, and $r_3 = (1 - i)\alpha$. At this point, we can use (22) to obtain (25).

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