THE

QUARTERLY JOURNAL

OF

PURE AND APPLIED

MATHEMATICS.

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VOL. XVIII.

London:
LONGMANS AND CO.,
PATERNOSTER ROW.

1882.

Example

$$x^2y^2 + (px + qy)\,xy + ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

is the general binodal quartic having nodes at A, B.

As an illustration, I give a solution of a problem set in a recent examination for Fellowships at S. John's.

"The tangents at the nodes of a binodal quartic meet the curve again in four points. These points and the two nodes lie on one conic."

The equations to the tangents at A are obtained simply by equating to zero the coefficient (a function of x) of the highest power of y, viz. they are in our case

$$x^2 + qx + b = 0.$$

The intersections with the curve are determined by this, and

$$(py + a)\,x^2 + 2hxy + 2gx + 2fy + c = 0.$$

Combining these we get

$$(py + a)\,(qx + b) = 2hxy + 2gx + 2fy + c.$$

By symmetry, the tangents at B would meet the quartic in points lying on the same curve, and its equation shows that is a conic passing through A and B. (For it is of the 2nd order, and there is a (1, 1) correspondence between x, y).

If, however, the quartic be such that $pq = 2h$, then the four points lie in one straight line.

Cambridge, *August*, 1881.

A SOLVIBLE CASE OF THE QUINTIC EQUATION.

By Professor CAYLEY.

THE roots of the general quintic equation

$$(a, b, c, d, e, f \between x, 1)^5 = 0,$$

may be taken to be

$$\begin{array}{l}
-\frac{b}{a} + \quad B + \quad C + \quad D + \quad E \\
-\,\text{,,} + \omega^4\,\text{,,} + \omega^3\,\text{,,} + \omega^2\,\text{,,} + \omega\;\text{,,} \\
-\,\text{,,} + \omega^3\,\text{,,} + \omega\;\text{,,} + \omega^4\,\text{,,} + \omega^2\,\text{,,} \\
-\,\text{,,} + \omega^2\,\text{,,} + \omega^4\,\text{,,} + \omega\;\text{,,} + \omega^3\,\text{,,} \\
-\,\text{,,} + \omega\;\text{,,} + \omega^2\,\text{,,} + \omega^3\,\text{,,} + \omega^4\,\text{,,},
\end{array}$$

where ω is an imaginary fifth root of unity; and if one of the four functions B, C, D, E is $=0$, say if $E=0$ (this implies of course a single relation between the coefficients), then the equation is solvible.

Writing $x=\xi-\frac{b}{a}$, we have

$$(a, b, c, d, e, f)\left(\xi-\frac{b}{a}, 1\right)^5=(a', 0, c', d', e', f'\between \xi, 1)^5,$$

where
$$\begin{aligned} a' &= a, \\ ac' &= ac - b^2, \\ a^2d' &= a^2d - 3abc + 2b^3, \\ a^3e' &= a^3e - 4a^2bd + 6ab^2c - 3b^4, \\ a^4f' &= a^4f - 5a^3be + 10ab^2d - 10ab^2c + 4b^5, \end{aligned}$$

and the roots of the new equation

$$(a', 0, c', d', e', f'\between \xi, 1)^5,$$

have the above-mentioned values, omitting therefrom the terms $-\frac{b}{a}$; we find without difficulty

$$2\frac{c'}{a}=-BE-CD,$$

$$2\frac{d'}{a}=-B^2D-BC^2-CE^2-D^2E,$$

$$\frac{e}{a}=-B^3C-B^2E^2+BCDE+BD^3+C^3E+C^2D^2-DE^3,$$

$$\frac{f'}{a}=-B^5+5B^3DE-5B^2C^2E-5B^2CD^2+5BC^3D+5BCE^3$$
$$-5BD^2E^2-C^5+5CD^3E-5CD^2E^2-D^5-E^5,$$

and hence, when $E=0$, we have

$$2\frac{c'}{a'}=-CD,$$

$$2\frac{d'}{a'}=-B^2D-BC^2,$$

$$\frac{e'}{a'}=-B^3C-BD^3-C^2D^2,$$

$$\frac{f'}{a'}=-B^5-5B^2CD^2+5BC^3D-C^5-D^5,$$

or, as these may be written,

$$-2\frac{c'}{a'} \qquad = CD,$$

$$-2\frac{d'}{a'} \qquad = B^2D + BC^2,$$

$$-\ \frac{e'}{a'} - 4\frac{c'^2}{a'^2} = B^3C - BD^3,$$

$$-\ \frac{f'}{a'} \qquad = B^5 + C^5 + D^5 - 10\frac{c'}{a'}(B^2D - BC^2),$$

equations which imply a single relation between the coefficients a', c', d', e', f'; supposing this satisfied, we may attend only to the first three equations; or, writing for convenience,

$$\gamma = -2\frac{c'}{a'}, \qquad = -\frac{2}{a^2}(ac - b^2),$$

$$\delta = -2\frac{d'}{a'}, \qquad = -\frac{2}{a^3}(a^2d - 3abc + 2b^3),$$

$$\theta = -\ \frac{e'}{a'} - 4\frac{c'^2}{a'^2}, \ = -\frac{1}{a^4}\{a^2(ae - 4bd + 3c^2) + (ac - b^2)\},$$

the equations are

$$\gamma = CD,$$

$$\delta = B(BD + C^2),$$

$$\theta = B(D^3 - B^2C).$$

The first equation gives $C = \frac{\gamma}{D}$, and substituting this value in the other two equations, we have

$$B^2D^3 + B\gamma^4 - \delta D^2 = 0,$$

$$B^3\gamma + BD^4 + \theta D = 0.$$

Eliminating B, the result is obtained in the form Det. $= 0$, where in the last colnmn of the determinant each term is divisible by D; and omitting this factor, the result is

$$\begin{vmatrix} & & D^3, & \gamma^2, & -\delta D \\ & D^3, & \gamma^2, & -\delta D^2, & \\ D^3, & \gamma^2, & -\delta D^2, & & \\ & \gamma, & 0, & -D^4, & \theta \\ \gamma, & 0, & -D^4, & \theta D, & \end{vmatrix} = 0,$$

and if, in order to develope the determinant, we consider it as a sum of products, each first factor being a minor composed out of columns 1 and 2, and the second factor being the complementary minor composed out of columns 3, 4, 5 (the several products being of course taken each with its proper sign), the expansion presents itself in the form

$$\begin{aligned} & D^3\gamma\,(-\theta\delta\gamma^2D^2+\delta^2D^7), \\ & -D^6\,(-\theta\gamma^2D^4+\delta D^9-\theta^2D^4) \\ & -\gamma D^3.-\delta D^2\,(\delta D^5-\theta\gamma^2) \\ & +\gamma^3\,(\gamma^2\delta D^5-\theta\delta D^5-\theta\gamma^4) \\ & -\gamma^2\,\delta^3D^5. \end{aligned}$$

Hence, collecting, and changing the sign of the whole expression, we obtain

$$\delta D^{15}-(2\gamma\delta^2+\gamma^2\theta+\theta^2)\,D^{10}+(-\gamma^3\delta+3\gamma\delta\theta+\delta^3)\,\gamma^2D^5+\gamma^7\theta=0,$$

a cubic equation for D^5. We have then as above $C=\dfrac{\gamma}{D}$, and B is given rationally as the common root of the foregoing quadric and cubic equations satisfied by B.

Substituting for γ, δ, θ their values in terms of the original coefficients, the equation for D^5 becomes

$$\begin{aligned} & 2\,(a^2d-3abc+2b^3)\,(aD)^{15} \\ & +\left\{\begin{array}{l} a^4\,(ae-4bd+3c^2)^2 \\ +a^2\,(ac-b^2)^2\,(ae-4bd+3c^2) \\ -16\,(ac-b^2)\,(a^2d-3abc+2b^3)^2 \end{array}\right\}(aD)^{10} \\ & +4\,(ac-b^2)^2\left\{\begin{array}{l} 28\,(ac-b^2)^3\,(a^2d-3abc+2b^3) \\ +12a^2(ac-b^2)(a^2d-3abc+2b^3)(ae-4bd+3c^2) \\ +\;8\qquad(a^2d-3abc+2b^3)^3 \end{array}\right\}(aD)^5 \\ & -128\,(ac-b^2)^7\,\{a^2\,(ae-4bd+3c^2)+(ac^2-b^2)^2\}=0, \end{aligned}$$

and the solution of the given quintic equation thus ultimately depends upon that of this cubic equation.
