EXAMPLE TO MOTIVATE A CORRECT DEFINITION OF LIMIT

Roger Chalkley, September 23, 2010

On page 803 of our textbook for Calculus 4 by Rowgawski (Calculus, Early Transcendentals — 1st edition, Freeman, New York, 2008), the author requires that the point approached be an element of the domain of the function and thereby introduces a definition that is inapplicable to the problems numbered 18, 23, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37 on pages 808-809. In order to correct this blunder, one may simply replace the statement

"Let $P = (a,\, b)$ be a point in the domain $\mathcal{D}$ of $f(x,\, y)$. Then"

in the second line of page 803 with the statement

"Let $P = (a,\, b)$ be a limit point of the domain $\mathcal{D}$ for the function $f(x,\, y)$. Then"

Rather than simply make this alteration as a suitable correction, I first chose to motivate a reasonable definition with the following example where the domain of the function is necessarily involved in a somewhat interesting manner.

With respect to

$$f(x, y) = \frac{x\,y}{\sqrt{x} + \sqrt{y}}, \quad \text{as a function of real variables } x \text{ and } y,$$

and our author's context where the domain is therefore

$$\mathcal{D} = \{(x,\, y) \mid\ x \geq 0,\ y \geq 0, \text{ and } (x,\, y) \neq (0,\, 0)\},$$

we see that $\mathcal{D}$ is described with respect to polar coordinates $x = r\,\cos\theta$, $y = r\,\sin\theta$ by the conditions $r > 0$ and $0 \leq \theta \leq \pi/2$. In particular, for $0 \leq \theta \leq \pi/4$, we have

$$\sqrt{x} + \sqrt{y} \geq \sqrt{r}\,\sqrt{\cos\frac{\pi}{4}} + 0 = \frac{\sqrt{r}}{\sqrt[4]{2}};$$

while, for $\pi/4 \leq \theta \leq \pi/2$, we see that

$$\sqrt{x} + \sqrt{y} \geq 0 + \sqrt{r}\,\sqrt{\sin\frac{\pi}{4}} = \frac{\sqrt{r}}{\sqrt[4]{2}}.$$

Thus, we obtain

$$0 \leq f(r\,\cos\theta,\, r\,\sin\theta) \leq \frac{r^2\,\sqrt[4]{2}}{\sqrt{r}}$$

and

$$\lim_{\substack{(x,\,y)\to(0,\,0)\\ (x,\,y)\text{ in }\mathcal{D}}} f(x,\, y) = 0.$$

If the use of the word limit point is deemed to be confusing for such a course, one could do as Serge Lang did (in his *A complete course in Calculus*) and use terminology like "a point close to the set $\mathcal{D}$" as a substitute for the concept of "a limit point for $\mathcal{D}$".