EFFICIENT POLYNOMIAL EVALUATION

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The old technique of "Synthetic Division" was always popular with students. Most of the Freshman College Algebra textbooks from about 1900 to 1950 presented it in a manner where specific examples of long division were modified to obtain an explanation of the procedure that was seen to work for particular examples. The procedure might have been more popular with teachers if it had been given a different name (e.g., the efficient polynomial-evaluation algorithm) and developed rigorously without any reference to division. For this purpose, I have developed the following explanation. It is a modification of ideas that have been around for hundreds of years.

For $n \geq 2$, let

$$f(X) \equiv a_0X^n + a_1X^{n-1} + a_2X^{n-2} + \cdots + a_{n-1}X + a_n \equiv \sum_{\nu=0}^{n} a_\nu X^{n-\nu} \tag{1}$$

denote a polynomial of degree n having complex coefficients and let r be a complex number. Then, according to the scheme

$$\begin{array}{cccccccc|l}
a_0 & a_1 & a_2 & a_3 & \ldots & a_{n-2} & a_{n-1} & a_n & \diagdown\underline{r} \\
 & r\,h_0 & r\,h_1 & r\,h_2 & \ldots & r\,a_{n-3} & r\,h_{n-2} & r\,h_{n-1} & \\
\hline
h_0 & h_1 & h_2 & h_3 & \ldots & h_{n-2} & h_{n-1} & h_n &
\end{array}$$

where

$$h_0 = a_0 \quad \text{and} \quad h_i = r\,h_{i-1} + a_i, \quad \text{for } 1 \leq i \leq n, \tag{2}$$

we see that the formula

$$h_i \equiv a_0\,r^i + a_1\,r^{i-1} + \cdots + a_i \equiv \sum_{\nu=0}^{i} a_\nu\,r^{i-\nu} \tag{3}$$

is valid for $i = 0$ and $i = 1$. Suppose that (3) is valid for some integer i that satisfies $1 \leq i \leq n-1$. Then, using (2) and (3), we find that

$$h_{i+1} \equiv r\,h_i + a_{i+1} \equiv \left[\sum_{\nu=0}^{i} a_\nu\,r^{i+1-\nu}\right] + a_{1+1} \equiv \sum_{\nu=0}^{i+1} a_\nu\,r^{i+1-\nu}.$$

Thus, (3) is valid for $0 \leq i \leq n$. In particular, for $i = n$, (3) yields

$$h_n \equiv \sum_{\nu=0}^{n} a_\nu\,r^{n-\nu} \equiv a_0\,r^n + a_1\,r^{n-1} + a_2\,r^{n-2} + \cdots + a_{n-1}\,r + a_n \equiv f(r).$$

Thus, h_n *is the value of the polynomial* $f(X)$ *when* r *is substituted for* X.

PROPOSITION. *In the preceding context, suppose that* $f(r) = 0$. *Then,* $f(X)$ *has the factorization*

$$f(X) \equiv \big(X - r\big)\big(h_0\,X^{n-1} + h_1\,X^{n-2} + \cdots + h_{n-1}\big) \equiv \big(X - r\big)\left[\sum_{\nu=0}^{n-1} h_\nu\,X^{n-1-\nu}\right]. \tag{4}$$

PROOF. We use (2), (1), and $f(r) = 0$ to deduce that

$$
\begin{aligned}
\bigl(X - r\bigr) \left[\sum_{\nu=0}^{n-1} h_\nu \, X^{n-1-\nu}\right] &\equiv \sum_{\nu=0}^{n-1} h_\nu \, X^{n-\nu} - \sum_{\nu=0}^{n-1} r \, h_\nu \, X^{n-1-\nu} \\
&\equiv \sum_{\nu=0}^{n-1} h_\nu \, X^{n-\nu} - \sum_{\nu=1}^{n} r \, h_{\nu-1} \, X^{n-\nu} \\
&\equiv a_0 \, X^n + \left[\sum_{\nu=1}^{n-1} \bigl(h_\nu - r \, h_{\nu-1}\bigr) X^{n-\nu}\right] - r \, h_{n-1} \\
&\equiv a_0 \, X^n + \left[\sum_{\nu=1}^{n-1} a_\nu \, X^{n-\nu}\right] - r \, h_{n-1} \\
&\equiv \left[\sum_{\nu=0}^{n} a_\nu \, X^{n-\nu}\right] + \bigl(-a_n - r \, h_{n-1}\bigr) \\
&\equiv f(X) - h_n \equiv f(X) - f(r) \equiv f(X).
\end{aligned}
$$

This completes the proof.