PRODUCT OF CIRCULANT MATRICES
and the Introduction of Group-Pattern Matrices

Roger Chalkley, November 20, 2021

The term *group-matrix* is a translation of Gruppenmatrix and refers to a special type of matrix that was introduced in research of Richard Dedekind in (click here, pages 423-425) and Georg Frobenius in (click here, page 1343). Its components are specialized variables.

The recent term *group-pattern matrix* of **Definition 1** below is more inclusive and has been needed to unify the subject. The convenient characterization of (5) was introduced for the 1976 paper (click here) and for the 1981 paper (click here). These papers would now be better titled *Diagonalizations of abelian-group-pattern matrices* and *Block-diagonalizations of group-pattern matrices.*

Next, we shall motivate **Definition 1** by employing the special case of it applied to circulant matrices.

Theorem 1. *Suppose that A and B are $n \times n$ circulant matrices having components in a ring $\mathfrak{R}$. Then, the matrix product of A and B is a circulant matrix and, when $\mathfrak{R}$ is a commutative ring, $BA = AB$.*

Proof. Let g be a generator for a cyclic group G of order n. Then, with e as the unit element of G and $g^n = e$, we obtain the multiplication table

	g	g^2	g^3	g^4	$\cdots$	g^n
g^{-1}	e	g	g^2	g^3	$\cdots$	g^{n-1}
g^{-2}	g^{n-1}	e	g	g^2	$\cdots$	g^{n-2}
g^{-3}	g^{n-2}	g^{n-1}	e	g	$\cdots$	g^{n-3}
g^{-4}	g^{n-3}	g^{n-2}	g^{n-1}	e	$\cdots$	g^{n-4}
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\ddots$	$\cdots$
g^{-n}	g	g^2	g^3	g^4	$\cdots$	e

for the elements of G. Consequently, there is a function α from G to $\mathfrak{R}$ such that the component $[A]_{r,s}$ of A in its rth row and sth column is given by

$$(1) \qquad [A]_{r,s} = \alpha\big(g^{-r} g^{s}\big), \quad \text{for } r,\, s = 1,\, 2,\, \ldots,\, n,$$

and there is a function β from G to $\mathfrak{R}$ such that the component $[B]_{r,s}$ of B in its rth row and sth column is given by

$$(2) \qquad [B]_{r,s} = \beta\big(g^{-r} g^{s}\big), \quad \text{for } r,\, s = 1,\, 2,\, \ldots,\, n.$$

Let γ be the function from G to $\mathfrak{R}$ defined by

$$(3) \qquad \gamma(x) = \sum_{\nu=1}^{n} \alpha\big(g^{\nu}\big)\, \beta\big(g^{-\nu} x\big), \quad \text{for each } x \text{ in } G.$$

We employ (1), (2), and (3) to see that the $(r,\,s)$-component of AB is given by

$$\begin{aligned}[AB]_{r,s} &= \sum_{k=1}^{n}[A]_{r,k}\,[B]_{k,s} = \sum_{k=1}^{n}\alpha\big(g^{k-r}\big)\,\beta\big(g^{s-k}\big) \\ &= \sum_{k=1}^{n}\alpha\big(g^{k-r}\big)\,\beta\big(g^{r-k}(g^{-r}g^{s})\big) \\ &= \sum_{\nu=1}^{n}\alpha(g^{\nu})\,\beta\big(g^{-\nu}(g^{-r}g^{s})\big) \\ &= \gamma\big(g^{-r}g^{s}\big), \quad \text{for } r,\, s = 1,\, 2,\, \ldots,\, n.\end{aligned} \tag{4}$$

Hence, AB is a circulant matrix.

Suppose that the ring $\mathfrak{R}$ is commutative. Then, for $r, s = 1,\, 2,\, \ldots,\, n$, we use (2), (1), (3), and (4) to obtain

$$\begin{aligned}[BA]_{r,s} &= \sum_{k=1}^{n}[B]_{r,k}\,[A]_{k,s} = \sum_{k=1}^{n}\beta\big(g^{k-r}\big)\,\alpha\big(g^{s-k}\big) \\ &= \sum_{k=1}^{n}\alpha\big(g^{s-k}\big)\,\beta\big(g^{k-s}(g^{-r}g^{s})\big) \\ &= \sum_{\nu=1}^{n}\alpha(g^{\nu})\,\beta\big(g^{-\nu}(g^{-r}g^{s})\big) \\ &= \gamma\big(g^{-r}g^{s}\big) = [AB]_{r,s}.\end{aligned}$$

Thus, we have $BA = AB$. This completes the proof. □

Definition 1. Let A be an $n \times n$ matrix having components in a set S and let G be a group of order n. Then, A is a *group-pattern matrix* for G and the list $\mathcal{L} = (g_1,\, g_2,\, \ldots,\, g_n)$ of the n elements in G if and only if there is a function σ from G to S such that the component $[A]_{r,s}$ of A in its rth row and sth column is given by

$$[A]_{r,s} = \sigma\big(g_r^{-1}g_s\big), \quad \text{for } r,\, s = 1,\, 2,\, \ldots,\, n. \tag{5}$$

To illustrate how various algebraic properties of circulant matrices can be transferred directly to group-pattern matrices, we supply a proof for the following result.

Theorem 2. Suppose that $n \times n$ matrices A and B with components in a ring $\mathfrak{R}$ are group-pattern matrices for a group G of order n and a list $\mathcal{L} = (g_1,\, g_2,\, \ldots,\, g_n)$ of the n elements in G. Then, the matrix product AB is a group-pattern matrix for G and $\mathcal{L}$. Moreover, when R is a commutative ring and G is abelian, $BA = AB$.

Proof. We apply (5) to see that there is a function α from G to $\mathfrak{R}$ such that the component $[A]_{r,s}$ of A in its rth row and sth column is given by

$$[A]_{r,s} = \alpha\big(g_r^{-1}g_s\big), \quad \text{for } r,\, s = 1,\, 2,\, \ldots,\, n, \tag{6}$$

and there is a function β from G to $\mathfrak{R}$ such that the component $[B]_{r,s}$ of B in its rth row and sth column is given by

$$[B]_{r,s} = \beta\big(g_r^{-1}g_s\big), \quad \text{for } r,\, s = 1,\, 2,\, \ldots,\, n. \tag{7}$$

Let γ denote the function from G to $\mathfrak{R}$ defined by

$$\gamma(x) = \sum_{\nu=1}^{n} \alpha\bigl(g_\nu\bigr)\, \beta\bigl(g_\nu^{-1} x\bigr), \quad \text{for each } x \text{ in } G. \tag{8}$$

We use (6), (7), and (8) to see that the $(r,\, s)$-component of AB is given by

$$\begin{aligned} [AB]_{r,s} &= \sum_{k=1}^{n} [A]_{r,k}\, [B]_{k,s} = \sum_{k=1}^{n} \alpha\bigl(g_r^{-1} g_k\bigr)\, \beta\bigl(g_k^{-1} g_s\bigr) \\ &= \sum_{k=1}^{n} \alpha\bigl(g_r^{-1} g_k\bigr)\ \beta\Bigl(\bigl(g_r^{-1} g_k\bigr)^{-1} \bigl(g_r^{-1} g_s\bigr)\Bigr) \\ &= \sum_{\nu=1}^{n} \alpha(g_\nu)\, \beta\bigl(g_\nu^{-1} (g_r^{-1} g_s)\bigr) = \gamma\bigl(g_r^{-1} g_s\bigr), \end{aligned} \tag{9}$$

for $r,\, s = 1,\, 2,\, \ldots,\, n$. Hence, AB is a group-pattern matrix for G and $\mathcal{L}$.

Suppose that the ring R is commutative and G is abelian. Then, for $r, s = 1,\, 2,\, \ldots,\, n$, we use (7), (6), (8), and (9) to obtain

$$\begin{aligned} [BA]_{r,s} &= \sum_{k=1}^{n} [B]_{r,k}\, [A]_{k,s} = \sum_{k=1}^{n} \beta\bigl(g_r^{-1} g_k\bigr)\, \alpha\bigl(g_k^{-1} g_s\bigr) \\ &= \sum_{k=1}^{n} \alpha\bigl(g_k^{-1} g_s\bigr)\, \beta\Bigl(\bigl(g_k^{-1} g_s\bigr)^{-1} \bigl(g_r^{-1} g_s\bigr)\Bigr) \\ &= \sum_{\nu=1}^{n} \alpha(g_\nu)\, \beta\Bigl(g_\nu^{-1} \bigl(g_r^{-1} g_s\bigr)\Bigr) \\ &= \gamma\bigl(g^{-r} g^{s}\bigr) = [AB]_{r,s}. \end{aligned}$$

Thus, we have $BA = AB$. This completes the proof. $\square$