CHAPTER 18

Suitable Context for Older Notation

The principal failing of the notation for (15.1), (15.3), and (15.6) that involves binomial coefficients was its role before 1989 in greatly hindering the discovery of suitable formulas for the coefficients of (15.6) corresponding to a change (15.5) of the independent variable. The details about this in Chapter 15 make it clear that the notation involving binomial coefficient should never have been adopted.

However, before we abandon that notation, an explanation should be given to explain why truly remarkable results like (17.4)–(17.5) of Edmund Laguerre and Georges-Henri Halphen were known to only a few mathematicians in 1989. Thus, we provide in Section 18.1 a previously missing precise context about invariants for equations like (15.1). However, to truly honor Laguerre and Halphen, their results should be presented in a form like that of Section 1.3 where binomial coefficients are avoided as a needless distraction.

18.1. Symbolism and terminology

In previous research about invariants for equations written as (15.1), instead of constructing semi-invariants and relative invariants from polynomials upon which algebraic operations can be performed and into which substitutions can be made, the semi-invariants and relative invariants were represented by functions without mention of substitutions. For example, the expression $C_2(z) - \big(C_1(z)\big)^2 - C_1^{(1)}(z)$ in the right member of (17.2), the expression inside the brackets of the right member for (17.3), and the expression in the right member of (17.4) were described as invariants.

For suitable notation, let $\mathfrak{R}_{m,1}$ be the ring of polynomials over $\mathbb{Q}$ in the variables

$$(18.1) \qquad \boldsymbol{W}_i^{(j)}, \quad \text{for } 1 \le i \le m \text{ and } j \ge 0;$$

set $\boldsymbol{W}_i \equiv \boldsymbol{W}_i^{(0)}$, for $1 \le i \le m$; and let $'$ denote the unique derivation for $\mathfrak{R}_{m,1}$ such that $\big(\boldsymbol{W}_i^{(j)}\big)' \equiv \boldsymbol{W}_i^{(j+1)}$, when $1 \le i \le m$ and $j \ge 0$. The constants of $\mathfrak{R}_{m,1}$ (i.e., the elements γ in $\mathfrak{R}_{m,1}$ having $\gamma' = 0$) are the elements of $\mathbb{Q}$. The *weight* of $\boldsymbol{W}_i^{(j)}$ is $i + j$; the *weight* of a nonzero element of $\mathbb{Q}$ is 0; and the *weight* of any nonzero monomial in $\mathfrak{R}_{m,1}$ is the sum of the weights of its factors. A polynomial in $\mathfrak{R}_{m,1}$ is said to be *isobaric* when it is nonzero and the weights of its nonzero terms are equal. The *weight* of an isobaric polynomial is the weight of any nonzero term.

For any polynomial $\widehat{\boldsymbol{R}}$ *in* $\mathfrak{R}_{m,1}$, *let* $\widehat{R}(z)$ *denote the function on* Ω *that is obtained by replacing each* $\boldsymbol{W}_i^{(j)}$ *of* $\widehat{\boldsymbol{R}}$ *with the corresponding* $C_i^{(j)}(z)$ *from* (15.1), *let* $\widehat{R}^*(z)$ *denote the function on* Ω *obtained by replacing each* $\boldsymbol{W}_i^{(j)}$ *of* $\widehat{\boldsymbol{R}}$ *with the corresponding* $C_i^{*(j)}(z)$ *from* (15.3), *and let let* $\widehat{R}^{**}(\zeta)$ *denote the function on* $.\Omega^{**}$ *obtained by replacing each* $\boldsymbol{W}_i^{(j)}$ *of* $\widehat{\boldsymbol{R}}$ *with the corresponding* $C_i^{**(j)}(\zeta)$ *from* (15.6).

For instance, in terms of the polynomial

$$\widehat{\boldsymbol{P}}_2 \equiv \boldsymbol{W}_2 - (\boldsymbol{W}_1)^2 - \boldsymbol{W}_1^{(1)}, \tag{18.2}$$

the identity (17.2) of James Cockle for $m \geq 2$ is $\widehat{P}_2^*(z) \equiv \widehat{P}_2(z)$, on Ω. Also, for

$$\widehat{\boldsymbol{Q}}_{m,2} \equiv \boldsymbol{W}_2 - \frac{(m-2)(3m-1)}{3(m-1)^2}\,(\boldsymbol{W}_1)^2 - \frac{2(m-2)}{3(m-1)}\boldsymbol{W}_1^{(1)}, \tag{18.3}$$

the identity (17.3) of Cockle is given by $\widehat{Q}_{m,2}^{**}(\zeta) \equiv \big(f'(\zeta)\big)^2 \widehat{Q}_{m,2}\big(f(\zeta)\big)$, on Ω^{**}. The identities (17.4) and (17.5) of Halphen are represented with the polynomial

$$\widehat{\boldsymbol{H}}_3 \equiv \boldsymbol{W}_3 - 3\boldsymbol{W}_2\boldsymbol{W}_1 + 2(\boldsymbol{W}_1)^3 - \tfrac{3}{2}\boldsymbol{W}_2^{(1)} + 3\boldsymbol{W}_1\boldsymbol{W}_1^{(1)} + \tfrac{1}{2}\boldsymbol{W}_1^{(2)}. \tag{18.4}$$

by $\widehat{H}_3^*(z) \equiv \widehat{H}_3(z)$, on Ω, and $\widehat{H}_3^{**}(\zeta) \equiv \big(f'(\zeta)\big)^3 \widehat{H}_3\big(f(\zeta)\big)$, on Ω^{**}. In fact, if the variables $\boldsymbol{W}_i^{(j)}$ of (18.1) are introduced so that they are related to the variables $\boldsymbol{w}_i^{(j)}$ for (1.13) by $\boldsymbol{w}_i^{(j)} \equiv \binom{m}{i}\boldsymbol{W}_i^{(j)}$, then $\boldsymbol{\mathcal{I}}_{m,1;\,3}$ in (1.13) yields $\boldsymbol{\mathcal{I}}_{m,1;\,3} \equiv \dbinom{m}{3}\widehat{\boldsymbol{H}}_3$.

DEFINITION 18.1. A polynomial $\widehat{\boldsymbol{R}}$ in $\mathfrak{R}_{m,1}$ is a *Cockle-semi-invariant of the first kind* for equations of the form (15.1) when it is not a constant and yields

$$\widehat{R}^*(z) \equiv \widehat{R}(z), \tag{18.5}$$

for each (15.1) and each transformation (15.2) of (15.1) into a corresponding (15.3).

DEFINITION 18.2. A polynomial $\widehat{\boldsymbol{R}}$ in $\mathfrak{R}_{m,1}$ is a *Cockle-semi-invariant of the second kind* for equations of the form (15.1) when it is not a constant and there is an integer s such that

$$\widehat{R}^{**}(\zeta) \equiv \big(f'(\zeta)\big)^s \widehat{R}\big(f(\zeta)\big), \tag{18.6}$$

for each (15.1) and each transformation (15.5) of (15.1) into a corresponding (15.6).

DEFINITION 18.3. A polynomial $\widehat{\boldsymbol{R}}$ in $\mathfrak{R}_{m,1}$ is a *Laguerre-Halphen relative invariant* for equations of the form (15.1) when it is both a Cockle-semi-invariant of the first kind and a Cockle-semi-invariant of the second kind for such equations.

As examples, we note that $\widehat{\boldsymbol{P}}_2$ in (18.2) is a Cockle-semi-invariant of the first kind; $\widehat{\boldsymbol{Q}}_{m,2}$ in (18.3) is a Cockle-semi-invariant of the second kind; and $\widehat{\boldsymbol{H}}_3$ in (18.4) is a Laguerre-Halphen relative invariant.

18.2. Our viewpoint abut the older Cockle-semi-invariants

To illustrate how the context of Section 18.1 can be employed, we begin by defining $\widehat{\boldsymbol{F}}_i$ and $\widehat{\boldsymbol{G}}_i$ in $\mathfrak{R}_{m,1}$ through

$$\widehat{\boldsymbol{F}}_0 \equiv 1, \quad \widehat{\boldsymbol{F}}_1 \equiv -\boldsymbol{W}_1, \quad \widehat{\boldsymbol{F}}_{i+1} \equiv \widehat{\boldsymbol{F}}_i^{(1)} + \widehat{\boldsymbol{F}}_1\widehat{\boldsymbol{F}}_i, \quad \text{for } i \geq 1, \tag{18.7}$$

and, with the introduction of $\boldsymbol{W}_0 \equiv 1$,

$$\widehat{\boldsymbol{G}}_i \equiv \sum_{j=0}^{i}\binom{i}{j}\widehat{\boldsymbol{F}}_{i-j}\boldsymbol{W}_j, \quad \text{when } 0 \leq i \leq m. \tag{18.8}$$

We have $\widehat{\boldsymbol{G}}_0 \equiv 1$, $\widehat{\boldsymbol{G}}_1 \equiv 0$, and $\widehat{\boldsymbol{G}}_2 \equiv \widehat{\boldsymbol{P}}_2$ in (18.2). The coefficients of $\widehat{\boldsymbol{F}}_i$ and $\widehat{\boldsymbol{G}}_i$ are polynomials in the variables $\boldsymbol{W}_j^{(k)}$ over $\mathbb{Q}$. They do not involve m.

To obtain $\widehat{G}_i(z)$ on Ω or $\widehat{G}_i^*(z)$ on Ω, we replace each $\boldsymbol{W}_j^{(k)}$ of $\widehat{\boldsymbol{G}}_i$ with the corresponding $C_j^{(k)}(z)$ from (15.1) or the corresponding $C_j^{*(k)}(z)$ from (15.3).

THEOREM 18.4. *For* $2 \le i \le m$ *and the equations* (15.1), $\widehat{\boldsymbol{G}}_i$ *is an isobaric Cockle-semi-invariant of the first kind having weight* i.

PROOF. For $2 \le i \le m$, we use (18.8) and (18.7) to see that the coefficient of $\boldsymbol{W}_i$ in $\widehat{\boldsymbol{G}}_i$ is 1 and therefore $\widehat{\boldsymbol{G}}_i$ is not a constant. Since (18.7) shows that $\widehat{\boldsymbol{F}}_i$ is an isobaric polynomial of weight i for $i \ge 0$, we apply (18.8) to conclude, for $2 \le i \le m$, that $\widehat{\boldsymbol{G}}_i$ is an isobaric polynomial of weight i.

For $0 \le i \le m$ and a transformation (15.2) of (15.1) into a corresponding (15.3), we employ (18.8), (15.4), and the identity

$$\binom{i}{j}\binom{j}{k} \equiv \binom{i}{k}\binom{i-k}{j-k}, \quad \text{for } 0 \le k \le j \le i,$$

to obtain

$$
\begin{aligned}
\text{(18.9)} \qquad \widehat{G}_i^*(z) &\equiv \sum_{j=0}^{i} \binom{i}{j} \widehat{F}_{i-j}^*(z)\, C_j^*(z) \\
&\equiv \sum_{j=0}^{i} \binom{i}{j} \widehat{F}_{i-j}^*(z) \sum_{k=0}^{j} \binom{j}{k} \frac{\rho^{(j-k)}(z)}{\rho(z)}\, C_k(z) \\
&\equiv \sum_{k=0}^{i} \binom{i}{k} C_k(z) \sum_{j=k}^{i} \binom{i-k}{j-k} \frac{\rho^{(j-k)}(z)}{\rho(z)}\, \widehat{F}_{i-j}^*(z) \\
&\equiv \sum_{k=0}^{i} \binom{i}{k} C_k(z) \sum_{\nu=0}^{i-k} \binom{i-k}{\nu} \frac{\rho^{(\nu)}(z)}{\rho(z)}\, \widehat{F}_{i-k-\nu}^*(z) \\
&\equiv \sum_{k=0}^{i} \binom{i}{k} C_k(z)\, S_{i-k}(z), \quad \text{on } \Omega,
\end{aligned}
$$

where

$$\text{(18.10)} \qquad S_\mu(z) \equiv \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} \frac{\rho^{(\nu)}(z)}{\rho(z)}\, \widehat{F}_{\mu-\nu}^*(z), \quad \text{on } \Omega \text{ for } \mu \ge 0.$$

We note that (18.10), (18.7), and (17.1) yield $S_0(z) \equiv \widehat{F}_0(z)$ and

$$S_1(z) \equiv \widehat{F}_1^*(z) + \frac{\rho^{(1)}(z)}{\rho(z)} \equiv -C_1^*(z) + \frac{\rho^{(1)}(z)}{\rho(z)} \equiv -C_1(z) \equiv \widehat{F}_1(z).$$

Let μ be an integer satisfying $\mu \ge 1$ such that $S_\mu(z) \equiv \widehat{F}_\mu(z)$. Then, we use (18.7), $\widehat{F}_\mu(z) \equiv S_\mu(z)$, $\widehat{F}_1(z) \equiv \rho^{(1)}(z)/\rho(z) + \widehat{F}_1^*(z)$, and (18.10) to verify that

$$
\begin{aligned}
\widehat{F}_{\mu+1}(z) &\equiv \widehat{F}_\mu^{(1)}(z) + \widehat{F}_1(z)\, \widehat{F}_\mu(z) \\
&\equiv S_\mu^{(1)} z) + \frac{\rho^{(1)}(z)}{\rho(z)}\, S_\mu(z) + \widehat{F}_1^*(z)\, S_\mu(z) \\
&\equiv \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} \frac{\rho^{(\nu+1)}(z)}{\rho(z)}\, \widehat{F}_{\mu-\nu}^*(z) - \frac{\rho^{(1)}(z)}{\rho(z)}\, S_\mu(z) + \frac{\rho^{(1)}(z)}{\rho(z)}\, S_\mu(z) \\
&\quad + \left(\sum_{\nu=0}^{\mu} \binom{\mu}{\nu} \frac{\rho^{(\nu)}(z)}{\rho(z)}\, \widehat{F}_{\mu-\nu}^{*(1)}(z) + \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} \frac{\rho^{(\nu)}(z)}{\rho(z)}\, \widehat{F}_1^*(z)\, \widehat{F}_{\mu-\nu}^*(z) \right)
\end{aligned}
$$

and, since the relation $\widehat{\boldsymbol{F}}_{i+1} \equiv \widehat{\boldsymbol{F}}_i^{(1)} + \widehat{\boldsymbol{F}}_1 \widehat{\boldsymbol{F}}_i$ in (18.7) is also valid for $i = 0$,

$$\begin{aligned}\widehat{F}_{\mu+1}(z) &\equiv \sum_{\nu=1}^{\mu+1} \binom{\mu}{\nu-1} \frac{\rho^{(\nu)}(z)}{\rho(z)}\, \widehat{F}^{*}_{\mu+1-\nu}(z) + \sum_{\nu=0}^{\mu} \binom{\mu}{\nu} \frac{\rho^{(\nu)}(z)}{\rho(z)}\, \widehat{F}^{*}_{\mu+1-\nu}(z)\\ &\equiv \sum_{\nu=0}^{\mu+1} \binom{\mu+1}{\nu} \frac{\rho^{(\nu)}(z)}{\rho(z)}\, \widehat{F}^{*}_{\mu+1-\nu}(z) \equiv S_{\mu+1}(z), \quad \text{on } \Omega.\end{aligned}$$

Thus, $S_\mu(z) \equiv \widehat{F}_\mu(z)$ is valid on Ω for $\mu \geq 0$. We replace $S_{i-k}(z)$ in (18.9) with $\widehat{F}_{i-k}(z)$ and compare the result with (18.8) to see, for any (15.1) and (15.2), that

$$\widehat{G}^{*}_{i}(z) \equiv \sum_{k=0}^{i} \binom{i}{k} C_k(z)\, \widehat{F}_{i-k}(z) \equiv \widehat{G}_i(z), \quad \text{on } \Omega \text{ when } 0 \leq i \leq m.$$

Hence, for $2 \leq i \leq m$, $\widehat{\boldsymbol{G}}_i$ is an isobaric Cockle-semi-invariant of the first kind having weight i. This completes the proof. □

This proof of Theorem 18.4 illustrates how the context of Section 18.1 can be applied. To connect it with the argument in Subsection 18.3.2 as the only one available for earlier researchers, we have the following result.

THEOREM 18.5. *For an equation* (15.1) *on Ω having order $m \geq 2$, repeated as*

$$y^{(m)}(z) + \sum_{i=1}^{m} \binom{m}{i} C_i(z)\, y^{(m-i)}(z) = 0,$$

suppose that $\rho_1(z) \not\equiv 0$ is a meromorphic function on a subregion $\mathcal{U}_1$ of Ω such that the substitution $y(z) = \rho_1(z)\, t(z)$ transforms the restriction to $\mathcal{U}_1$ of (15.1) *into*

$$t^{(m)}(z) + \sum_{i=2}^{m} \binom{m}{i} d_i(z)\, t^{(m-i)}(z) = 0, \quad \text{on } \mathcal{U}_1 \text{ with } d_1(z) \equiv 0. \tag{18.11}$$

Then, for $1 \leq i \leq m$, $d_i(z)$ of (18.11) *is given by $d_i(z) \equiv \widehat{G}_i(z)$, on $\mathcal{U}_1$.*

PROOF. For the indicated transformation of (15.1) on $\mathcal{U}_1$ into (18.11) on $\mathcal{U}_1$, we find that (15.4) and (18.7) yield

$$d_1(z) \equiv \frac{\rho_1^{(1)}(z)}{\rho_1(z)} + C_1(z) \equiv 0 \quad \text{and} \quad \widehat{F}_1(z) \equiv -C_1(z) \equiv \frac{\rho_1^{(1)}(z)}{\rho_1(z)}, \quad \text{on } \mathcal{U}_1. \tag{18.12}$$

We use (18.7) and (18.12) to see that the formula

$$\widehat{F}_k(z) \equiv \frac{\rho_1^{(k)}(z)}{\rho_1(z)}, \quad \text{on } \mathcal{U}_1, \tag{18.13}$$

is true for $k = 0$ and $k = 1$. In terms of any positive integer k for which (18.13) is valid, we observe that (18.7) and (18.13) yield

$$\begin{aligned}\widehat{F}_{k+1}(z) &\equiv \widehat{F}^{(1)}_k(z) + \widehat{F}_1(z)\, \widehat{F}_k(z)\\ &\equiv \frac{\rho_1^{(k+1)}(z)}{\rho_1(z)} - \frac{\rho_1^{(k)}(z)\, \rho_1^{(1)}(z)}{\big(\rho_1(z)\big)^2} + \frac{\rho_1^{(1)}(z)}{\rho_1(z)} \frac{\rho_1^{(k)}(z)}{\rho_1(z)} \equiv \frac{\rho_1^{(k+1)}(z)}{\rho_1(z)}.\end{aligned}$$

Thus, (18.13) is true for $k \geq 0$. Using (15.4), (18.13), and (18.8), we notice that the substitution $y(z) = \rho_1(z)\, v(z)$ transforms the restriction to $\mathcal{U}_1$ of (15.1) into the equation (18.11) on $\mathcal{U}_1$ having

$$d_i(z) \equiv \sum_{j=0}^{i} \binom{i}{j} \frac{\rho_1^{(i-j)}(z)}{\rho_1(z)}\, C_j(z) \equiv \sum_{j=0}^{i} \binom{i}{j} \widehat{F}_{i-j}(z)\, C_j(z) \equiv \widehat{G}_i(z),$$

for $2 \leq i \leq m$. This completes the proof. □

18.3. Original introduction of $\widehat{G}_i(z)$, $\widehat{G}_i^*(z)$, and $\widehat{G}_i^*(z) \equiv \widehat{G}_i(z)$

Let (15.1) be an equation of order $m \geq 2$ on Ω and let $y(z) = \rho(z)\, v(z)$ be a substitution as described for (15.2) that transforms (15.1) on Ω into (15.3) on Ω. Here, we are to ignore the content of Sections 18.1 and 18.2 in order to view the way that Georges-Henri Halphen in [**32**] and Andrew Forsyth in [**28**] introduced the Cockle-semi-invariants of the first kind that were essential for their constructions.

18.3.1. Halphen canonical form for (15.1)**.** Let $\rho_1(z)$ be a meromorphic function on a subregion $\mathcal{U}_1$ of Ω such that $\rho_1^{(1)}(z) + C_1(z)\,\rho_1(z) \equiv 0$ on $\mathcal{U}_1$ and $\rho_1(z) \not\equiv 0$. Then, we use (15.4) to see that the substitution $y(z) = \rho_1(z)\, t(z)$ transforms the restriction to $\mathcal{U}_1$ of (15.1) into the equation on $\mathcal{U}_1$ given by

$$t^{(m)}(z) + \sum_{i=2}^{m} \binom{m}{i} \widehat{G}_i(z)\, t^{(m-i)}(z) = 0, \quad \text{with } \widehat{G}_1(z) \equiv 0, \tag{18.14}$$

where explicit expressions for $\widehat{G}_2(z), \ldots, \widehat{G}_m(z)$ are obtained by substituting

$$\frac{\rho_1^{(1)}(z)}{\rho_1(z)} \equiv -C_1(z), \quad \frac{\rho_1^{(2)}(z)}{\rho_1(z)} \equiv -C_1^{(1)}(z) + \left(\frac{\rho_1^{(1)}(z)}{\rho_1(z)}\right)^2 \equiv -C_1^{(1)}(z) + \big(C_1(z)\big)^2,$$

and

$$\frac{\rho_1^{(k+1)}(z)}{\rho_1(z)} \equiv \left(\frac{\rho_1^{(k)}(z)}{\rho_1(z)}\right)' + \frac{\rho_1^{(k)}(z)}{\rho_1(z)}\,\frac{\rho_1^{(1)}(z)}{\rho_1(z)}, \quad \text{for } k \geq 2,$$

into (15.4) to obtain $\widehat{G}_1(z) \equiv 0$, $\widehat{G}_2(z) \equiv C_2(z) - \big(C_1(z)\big)^2 - C_1^{(1)}(z), \ldots$. But, the coefficients of (18.14) are defined on all of Ω and they are uniquely specified by the given (15.1). In this way, apart from the selection of the variable t, the equation (15.1) on Ω uniquely determines (18.14) on Ω as its *Halphen canonical form.*

18.3.2. Deduction of $\widehat{G}_i^*(z) \equiv \widehat{G}_i(z)$. Just as (15.1) on Ω specifies (18.14) on Ω as its Halphen canonical form, the equation (15.3) on Ω specifies

$$t^{(m)}(z) + \sum_{i=2}^{m} \binom{m}{i} \widehat{G}_i^*(z)\, t^{(m-i)}(z) = 0, \quad \text{with } \widehat{G}_1^*(z) \equiv 0, \tag{18.15}$$

on Ω as its Halphen canonical form where $\widehat{G}_2^*(z) \equiv C_2^*(z) - \big(C_1^*(z)\big)^2 - C_1^{*(1)}(z)$, and so on. For $1 \leq i \leq m$, $\widehat{G}_i^*(z)$ was regarded as obtained from $\widehat{G}_i(z)$ by replacing in $\widehat{G}_i(z)$ each $C_j^{(k)}(z)$ from (15.1) with the corresponding $C_j^{*(k)}(z)$ from (15.3).

With reference to $\mathcal{U}_1$ for the local transformation $y(z) = \rho_1(z)\, t(z)$ of (15.1) on $\mathcal{U}_1$ into (18.14) on $\mathcal{U}_1$, we observe that the substitution $v(z) = \big(1/\rho(z)\big)y(z)$ transforms the restriction to $\mathcal{U}_1$ of (15.3) into the restriction to $\mathcal{U}_1$ of (15.1). Hence, the substitution $v(z) = \big(\rho_1(z)/\rho(z)\big)t(z)$ transforms the restriction to $\mathcal{U}_1$ of (15.3) into the restriction to $\mathcal{U}_1$ of (18.14). Consequently, both (18.14) and (18.15) are Halphen canonical forms for (15.3). *This requires $\widehat{G}_i^*(z) \equiv \widehat{G}_i(z)$, for $2 \leq i \leq m$.*

Since Georges-Henri Halphen in [**32**] of 1884, Andrew Forsyth in [**28**] of 1888, and other researchers did not employ polynomials into which substitutions from (15.1) or (15.3) or (15.6) could be performed, the function $\widehat{G}_i(z)$, for $2 \leq i \leq m$, was referred to as *an isobaric semi-invariant of the first kind having weight i.*

Sections 18.1 and 18.2 provide clarification. See Theorems 18.4 and 18.5.

18.4. Results of Forsyth in the context for Sections 18.1 and 18.2

Andrew Forsyth employed infinitesimal transformations in [**28**, pages 398–401] of 1888 to derive a necessary structure for Laguerre-Halphen relative invariants of weights $s = 3, 4, 5, 6, 7$ for equations (15.1) of order $m \geq s$. The deduction for the weight $s = 3$ in [**28**, page 398, (14)] corresponds to the notation

$$\widehat{\Theta}_3(z) \equiv \widehat{G}_3(z) - (3/2)\,\widehat{G}_2^{(1)}(z), \quad \text{for equations (15.1) of order } m \geq 3. \tag{18.16}$$

The right member of (18.16) is equal to the right member of (17.4) and was known to Georges-Henri Halphen in [**32**] of 1884. That it does not involve m was noted on page 169. In that regard, the notation of Sections 18.1 and 18.2 yields the identity

$$\widehat{\boldsymbol{\Theta}}_3 \equiv \widehat{\boldsymbol{G}}_3 - (3/2)\,\widehat{\boldsymbol{G}}_2^{(1)} \equiv \widehat{\boldsymbol{H}_3}, \tag{18.17}$$

where $\widehat{\boldsymbol{H}_3}$ is given by (18.4).

With respect to the notation of Section 18.1 on page 171, we use (18.8) to see that the polynomials of interest that correspond to the formulas for $\widehat{\Theta}_4(z)$, $\widehat{\Theta}_5(z)$, $\widehat{\Theta}_6(z)$, and $\widehat{\Theta}_7(z)$ in [**28**, pages 399–401, (15), (16),(17),(18)] are given by

$$\widehat{\boldsymbol{\Theta}}_4 \equiv \widehat{\boldsymbol{G}}_4 - 2\,\widehat{\boldsymbol{G}}_3^{(1)} + (6/5)\,\widehat{\boldsymbol{G}}_2^{(2)} - \frac{3(5m+7}{5(m+1)}\,\big(\widehat{\boldsymbol{G}}_2\big)^2, \tag{18.18}$$
$$\text{for equations (15.1) of order } m \geq 4,$$

$$\widehat{\boldsymbol{\Theta}}_5 \equiv \widehat{\boldsymbol{G}}_5 - \frac{5}{2}\,\widehat{\boldsymbol{G}}_4^{(1)} + \frac{15}{7}\,\widehat{\boldsymbol{G}}_3^{(2)} - \frac{5}{7}\,\widehat{\boldsymbol{G}}_2^{(3)} - \frac{10(7m+13)}{7(m+1)}\,\widehat{\boldsymbol{G}}_2\,\widehat{\boldsymbol{\Theta}}_3, \tag{18.19}$$
$$\text{for equations (15.1) of order } m \geq 5,$$

$$\begin{aligned} \widehat{\boldsymbol{\Theta}}_6 \equiv{}& \widehat{\boldsymbol{G}}_6 - 3\,\widehat{\boldsymbol{G}}_5^{(1)} + (10/3)\,\widehat{\boldsymbol{G}}_4^{(2)} - (5/3)\,\widehat{\boldsymbol{G}}_3^{(3)} + (5/14)\,\widehat{\boldsymbol{G}}_2^{(4)} \\ &+ \frac{30(7m^2+28m+25)}{7(m+1)^2}\,\big(\widehat{\boldsymbol{G}}_2\big)^3 + \frac{5(7m+8)}{14(m+1)}\,\Big(\widehat{\boldsymbol{G}}_2^{(1)}\Big)^2 \\ &- 5\,\frac{3m+7}{m+1}\,\widehat{\boldsymbol{G}}_2\Big(\widehat{\boldsymbol{G}}_4 - 2\,\widehat{\boldsymbol{G}}_3^{(1)} + \frac{2(14m+31)}{7(3m+7)}\,\widehat{\boldsymbol{G}}_2^{(2)}\Big), \end{aligned} \tag{18.20}$$
$$\text{for equations (15.1) of order } m \geq 6,$$

and

$$\begin{aligned} \widehat{\boldsymbol{\Theta}}_7 \equiv{}& \widehat{\boldsymbol{G}}_7 - \frac{7}{2}\,\widehat{\boldsymbol{G}}_6^{(1)} + \frac{105}{22}\,\widehat{\boldsymbol{G}}_5^{(2)} - \frac{35}{11}\,\widehat{\boldsymbol{G}}_4^{(3)} + \frac{35}{33}\widehat{\boldsymbol{G}}_3^{(4)} - \frac{7}{44}\,\widehat{\boldsymbol{G}}_2^{(5)} \\ &- \frac{7\,\widehat{\boldsymbol{G}}_2}{11(m+1)}\left[\begin{aligned} &(3/2)(11m+31)\Big(2\,\widehat{\boldsymbol{G}}_5 - 5\,\widehat{\boldsymbol{G}}_4^{(1)}\Big) \\ &+ 5(15m+41)\,\widehat{\boldsymbol{G}}_3^{(2)} - 15(2m+5)\,\widehat{\boldsymbol{G}}_2^{(3)} \end{aligned}\right] \\ &- \frac{7(3m+4)}{11(m+1)}\,\Big[3\,\widehat{\boldsymbol{G}}_2^{(2)}\Big(\widehat{\boldsymbol{G}}_3 + \widehat{\boldsymbol{G}}_2^{(1)}\Big) - 5\,\widehat{\boldsymbol{G}}_2^{(1)}\widehat{\boldsymbol{G}}_3^{(1)}\Big] \\ &+ \frac{21(55m^2+288m+329)}{11(m+1)^2}\,\big(\widehat{\boldsymbol{G}}_2\big)^2\widehat{\boldsymbol{\Theta}}_3, \end{aligned} \tag{18.21}$$
$$\text{for equations (15.1) of order } m \geq 7,$$

except that: the denominator $11(m+1)^2$ appearing in the last fraction of (18.21) is a correction for the denominator $22(m+1)^2$ that [**28**, page 401, (18)] would give. For details about that misprint, see [**19**, page 79].

The restriction to infinitesimal transformations is removed in Theorem 18.7.

18.4.1. Deduction for $\widehat{\boldsymbol{G}}_i^{(k)}$. For any polynomial $\widehat{\boldsymbol{P}}$ in $\mathfrak{R}_{m,1}$ of page 171 and the derivation $'$ for $\mathfrak{R}_{m,1}$, we write $\widehat{\boldsymbol{P}}^{(0)} \equiv \widehat{\boldsymbol{P}}$, $\widehat{\boldsymbol{P}}^{(1)} \equiv \big(\widehat{\boldsymbol{P}}^{(0)}\big)'$, $\widehat{\boldsymbol{P}}^{(2)} \equiv \big(\widehat{\boldsymbol{P}}^{(1)}\big)'$, $\ldots$. Thus, for any $k \geq 0$, we use the notation $\widehat{\boldsymbol{P}}^{(k)}$ for the polynomial in $\mathfrak{R}_{m,1}$ obtained from $\widehat{\boldsymbol{P}}$ by repeatedly applying k times the derivation $'$ defined for $\mathfrak{R}_{m,1}$.

We recall that $\widehat{P}(z)$ designates the function on Ω obtained from $\widehat{\boldsymbol{P}}$ by replacing each $\boldsymbol{W}_i^{(j)}$ in $\widehat{\boldsymbol{P}}$ with the corresponding $C_i^{(j)}(z)$ from (15.1). However, due to properties of the derivation $'$ for $\mathfrak{R}_{m,1}$, we see that the function $\big(\widehat{P}^{(k)}\big)(z)$ on Ω that is obtained from $\widehat{\boldsymbol{P}}^{(k)}$ by replacing each $\boldsymbol{W}_i^{(j)}$ in $\widehat{\boldsymbol{P}}^{(k)}$ with the corresponding $C_i^{(j)}(z)$ from (15.1) is equal to the kth derivative with respect to z of $\widehat{P}(z)$. Similarly, the function $\big(\widehat{P}^{(k)}\big)^*(z)$ on Ω obtained from $\widehat{\boldsymbol{P}}^{(k)}$ by replacing each $\boldsymbol{W}_i^{(j)}$ in $\widehat{\boldsymbol{P}}^{(k)}$ with the corresponding $C_i^{*(j)}(z)$ from (15.3) is equal to the kth derivative with respect to z of the function $\widehat{P}^*(z)$ obtained from $\widehat{\boldsymbol{P}}$ by replacing each $\boldsymbol{W}_i^{(j)}$ in $\widehat{\boldsymbol{P}}$ with the corresponding $C_i^{*(j)}(z)$ from (15.3).

PROPOSITION 18.6. *For $2 \leq i \leq m$ and $k \geq 0$, $\widehat{\boldsymbol{G}}_i^{(k)}$ is an isobaric Cockle-semi-invariant of the first kind having weight $i+k$.*

PROOF. For $2 \leq i \leq m$ and $k \geq 0$, we use (18.8) to see that the coefficient of $\boldsymbol{W}_i^{(k)}$ in $\widehat{\boldsymbol{G}}_i^{(k)}$ is 1 and $\widehat{\boldsymbol{G}}_i^{(k)}$ is an isobaric polynomial of weight $i+k$. Theorem 18.4 on page 173 yields $\widehat{G}_i^*(z) \equiv \widehat{G}_i(z)$, on Ω, from which we deduce

$$\big(\widehat{G}_i^{(k)}\big)^*(z) \equiv \frac{d^k}{dz^k}\,\widehat{G}_i^*(z) \equiv \frac{d^k}{dz^k}\,\widehat{G}_i(z) \equiv \big(\widehat{G}_i^{(k)}\big)(z), \quad \text{on } \Omega. \tag{18.22}$$

We compare (18.22) with (18.6) of Definition 18.1 on page 172 to complete the proof. ☐

18.4.2. Properties of $\widehat{\boldsymbol{\Theta}}_3$, $\widehat{\boldsymbol{\Theta}}_4$, $\widehat{\boldsymbol{\Theta}}_5$, $\widehat{\boldsymbol{\Theta}}_6$, and $\widehat{\boldsymbol{\Theta}}_7$. We know that $\widehat{\boldsymbol{\Theta}}_3$ in (18.17) is a Laguerre-Halphen relative invariant for equations (15.1) of order $m \geq 3$ because $\widehat{\boldsymbol{H}}_3$ in (18.4) on page 172 has that property with respect to Definition 18.3.

THEOREM 18.7. *For $3 \leq s \leq 7$, $\widehat{\boldsymbol{\Theta}}_s$ is a Laguerre-Halphen relative invariant of weight s for the equations* (15.1) *of order $m \geq s$.*

PROOF. Let s satisfy $3 \leq s \leq 7$. We use (18.17)–(18.21) to see that the coefficient of $\boldsymbol{W}_s$ in $\widehat{\boldsymbol{\Theta}}_s$ is equal to the coefficient 1 of $\boldsymbol{W}_s$ in $\widehat{\boldsymbol{G}}_s$. Since Definition 18.1 on page 172 shows that a nonzero sum of Cockle-semi-invariants of the first kind is a Cockle-semi-invariant of the first kind, we apply Proposition 18.6 to conclude that $\widehat{\boldsymbol{\Theta}}_s$ is an isobaric Cockle-semi-invariant of the first kind having weight s.

We establish in Section 18.5 that: when an equation (15.1) on Ω is transformed by $z = f(\zeta)$ of (15.5) into a corresponding equation (15.6) on Ω^{**}, the identity

$$\widehat{\Theta}_s^{**}(\zeta) - \big(f'(\zeta)\big)^s\,\widehat{\Theta}_s\big(f(\zeta)\big) \equiv 0, \quad \text{on } \Omega^{**} \text{ for } 3 \leq s \leq 7, \tag{18.23}$$

is valid, where $\widehat{\Theta}_s(z)$ on Ω and $\widehat{\Theta}_s^{**}(\zeta)$ on Ω^{**} are obtained by replacing each $\boldsymbol{W}_i^{(j)}$ in $\widehat{\boldsymbol{\Theta}}_s$ with the corresponding $C_i^{(j)}(z)$ from (15.1) and $C_i^{**(j)}(\zeta)$ from (15.6). In view of (18.23) and Definitions 18.2–18.3, we conclude that $\widehat{\boldsymbol{\Theta}}_s$ is a Laguerre-Halphen relative invariant of weight s. ☐

18.5. Computer-algebra verification of (18.23)

After selecting a version of *Mathematica* from [**55, 56, 57, 58, 59**] as the system, we recall from page 167 that the evaluations of

```
Ce[m_,0][z_] := 1

alpha[0,j_][zeta_] := 1

alpha[i_,j_][zeta_] := ( Sum[alpha[i-1,k]'[zeta]
  -(i-1+k)(f''[zeta]/f'[zeta])*
  alpha[i-1,k][zeta],{k,1,j}] ) /; i >= 1

beta[m_,r_,s_][zeta_]:=(Product[(m-s-k+1),{k,1,r}]/
  Product[(s+k),{k,1,r}])*alpha[r,s][zeta]

CeSS[m_,i_][zeta_] := Sum[beta[m,i-j,m-i][zeta]*
  (f'[zeta])^j*Ce[m, j][f[zeta]],{j,0,i}]
```

provide representations for the coefficients $C_1^{**}(\zeta)$, $C_2^{**}(\zeta)$, $\ldots$ of (15.6). We use (18.7) and (18.8) to see that the evaluations of

```
F[0][z_] := 1

F[1][z_] := - Ce[m,1][z]

F[i_][z_] := F[i-1]'[z]+F[1][z]*F[i-1][z] /; i >= 2

G[i_][z_] := Sum[Binomial[i,j]*F[i-j][z]*Ce[m,j][z],{j,0,i}]

FSS[0][zeta_] := 1

FSS[1][zeta_] := - CeSS[m,1][zeta]

FSS[i_][zeta_] :=
  FSS[i-1]'[zeta]+FSS[1][zeta]*FSS[i-1][zeta] /; i >= 2

GSS[i_][zeta_] :=
  Sum[Binomial[i,j]*FSS[i-j][zeta]*CeSS[m,j][zeta],{j,0,i}]
```

provide representations for the function $\widehat{G}_i(z)$ on Ω and $\widehat{G}_i^{**}(\zeta)$ on Ω^{**} obtained from $\widehat{\boldsymbol{G}}_i$ by replacing each $\boldsymbol{W}_j$ in $\widehat{\boldsymbol{G}}_i$ with the corresponding $C_j(z)$ from (15.1) or $C_j^{**}(\zeta)$ from (15.6). In view of (18.17)–(18.21), we observe that the evaluations of

```
Theta[3][z_] := ( G[3][z]-(3/2)G[2]'[z] )

ThetaSS[3][zeta_] := ( GSS[3][zeta]-(3/2)GSS[2]'[zeta] )

Theta[4][z_] := ( G[4][z]-2G[3]'[z]+(6/5)G[2]''[z]
  -(3/5)((5m+7)/(m+1))G[2][z]^2 )
```

```
ThetaSS[4][zeta_] := ( GSS[4][zeta]-2GSS[3]'[zeta]
  +(6/5)GSS[2]''[zeta]-(3/5)((5m+7)/(m+1))GSS[2][zeta]^2 )

Theta[5][z_] := ( G[5][z]-(5/2)G[4]'[z]+(15/7)G[3]''[z]
  -(5/7)G[2]'''[z]-(10/7)((7m+13)/(m+1))G[2][z]*Theta[3][z] )

ThetaSS[5][zeta_] := ( GSS[5][zeta]-(5/2)GSS[4]'[zeta]
  +(15/7)GSS[3]''[zeta]-(5/7)GSS[2]'''[zeta]
  -(10/7)((7m+13)/(m+1))GSS[2][zeta]*ThetaSS[3][zeta] )

Theta[6][z_] := ( G[6][z]-3G[5]'[z]+(10/3)G[4]''[z]
  -(5/3)G[3]'''[z]+(5/14)G[2]''''[z]
  +(30/7)G[2][z]^3*((7m^2+28m+25)/(m+1)^2)
  +(5/14)((7m+8)/(m+1))G[2]'[z]^2
  -5((3m+7)/(m+1))G[2][z]*
     (G[4][z]-2G[3]'[z]+(2/7)((14m+31)/(3m+7))G[2]''[z]) )

ThetaSS[6][zeta_] := ( GSS[6][zeta]-3GSS[5]'[zeta]
  +(10/3)GSS[4]''[zeta]-(5/3)GSS[3]'''[zeta]
  +(5/14)GSS[2]''''[zeta]
  +(30/7)GSS[2][zeta]^3*((7m^2+28m+25)/(m+1)^2)
  +(5/14)((7m+8)/(m+1))GSS[2]'[zeta]^2
  -5((3m+7)/(m+1))GSS[2][zeta]*(GSS[4][zeta]
     -2GSS[3]'[zeta]+(2/7)((14m+31)/(3m+7))GSS[2]''[zeta]) )

Theta[7][z_] := ( G[7][z]-(7/2)G[6]'[z]+(105/22)G[5]''[z]
  -(35/11)G[4]'''[z]+(35/33)G[3]''''[z]-(7/44)G[2]'''''[z]
  -(7/11)(G[2][z]/(m+1))*( (3/2)(11m+31)(2G[5][z]-5G[4]'[z])
      +5(15m+41)G[3]''[z]-15(2m+5)G[2]'''[z] )
  -(7/11)((3m+4)/(m+1))*( 3G[2]''[z]( G[3][z]+G[2]'[z] )
      - 5G[2]'[z]*G[3]'[z] )
  +G[2][z]^2*Theta[3][z]*((1155m^2+6048m+6909)/(11(m+1)^2)) )

ThetaSS[7][zeta_] := ( GSS[7][zeta]-(7/2)GSS[6]'[zeta]
  +(105/22)GSS[5]''[zeta]-(35/11)GSS[4]'''[zeta]
  +(35/33)GSS[3]''''[zeta]-(7/44)GSS[2]'''''[zeta]
  -(7/11)(GSS[2][zeta]/(m+1))*( (3/2)(11m+31)(2GSS[5][zeta]
     -5GSS[4]'[zeta])+5(15m+41)GSS[3]''[zeta]
     -15(2m+5)GSS[2]'''[zeta] )
  - (7/11)((3m+4)/(m+1))( 3GSS[2]''[zeta]*
     (GSS[3][zeta]+GSS[2]'[zeta])-5GSS[2]'[zeta]*GSS[3]'[zeta])
  +GSS[2][zeta]^2*ThetaSS[3][zeta]*
     (21(55m^2+288m+329)/(11(m+1)^2)) )
```

represent the functions $\widehat{\Theta}_s(z)$ on Ω and $\widehat{\Theta}_s^{**}(\zeta)$ on Ω^{**}, for $3 \le s \le 7$, that appear in (18.23) on page 177. Since the evaluation for each of

```
FullSimplify[ ThetaSS[3][zeta] - f'[zeta]^3*Theta[3][f[zeta]] ]

FullSimplify[ ThetaSS[4][zeta] - f'[zeta]^4*Theta[4][f[zeta]] ]

FullSimplify[ ThetaSS[5][zeta] - f'[zeta]^5*Theta[5][f[zeta]] ]

FullSimplify[ ThetaSS[6][zeta] - f'[zeta]^6*Theta[6][f[zeta]] ]

FullSimplify[ ThetaSS[7][zeta] - f'[zeta]^7*Theta[7][f[zeta]] ]
```

is zero, we conclude that (18.23) is valid. A *Mathematica* notebook containing the preceding evaluations can be downloaded from

`http://homepages.uc.edu/~chalklr/Chapter-18.html`

with the Google browser *Chrome*. It illustrates well the technique of Chapter 17.

18.6. Several observations

A different argument to verify Theorem 18.7 was employed for [**19**, page 79]. There, after finding explicit formulas for all of the basic relative invariants of the equations (15.9), we used computer algebra with the substitution $\boldsymbol{w}_i^{(j)} = \binom{m}{i}\boldsymbol{W}_i^{(j)}$ in the basic relative invariants $\boldsymbol{\mathcal{I}}_{m,1;\,3}, \ldots, \boldsymbol{\mathcal{I}}_{m,1;\,7}$ to verify that

$$\boldsymbol{\mathcal{I}}_{m,1;\,s} \equiv \binom{m}{s}\widehat{\boldsymbol{\Theta}}_s, \quad \text{for } 3 \le s \le 7 \text{ and } m \ge s, \tag{18.24}$$

where $\widehat{\boldsymbol{\Theta}}_3, \ldots, \widehat{\boldsymbol{\Theta}}_7$ are given by (18.17)–(18.21). The properties of $\widehat{\boldsymbol{\Theta}}_s$ as a Laguerre-Halphen relative invariant for the equations (15.1) then follow from properties of $\boldsymbol{\mathcal{I}}_{m,1;\,s}$ as a relative invariant for the equations (15.9).

The formulas for $\widehat{\Theta}_3(z)$, $\widehat{\Theta}_4(z)$, $\widehat{\Theta}_5(z)$, $\widehat{\Theta}_6(z)$, and $\widehat{\Theta}_7(z)$ in [**28**, pages 398–401] and their rewritten versions appearing in [**8**, page 235] did not lead to general results. Francesc Brioschi introduced errors when he rewrote $\widehat{\Theta}_7(z)$ for [**8**, page 235] of 1891 and those errors were copied in the expression for $\widehat{\Theta}_7(z)$ that Ludwig Schlesinger included in [**47**, page 196] of 1897.

18.7. Computer-algebra verification of (18.24)

We continue with the *Mathematica* notebook that was begun on page 178 and includes the sixteen commands of page 178, the seven commands of page 179, and the five commands above. At this point, the evaluations of `Theta[3][z]`, `Theta[4][z]`, `Theta[5][z]`, `Theta[6][z]`, `Theta[7][z]` are representations for the functions $\widehat{\Theta}_3(z)$, $\widehat{\Theta}_4(z)$, $\widehat{\Theta}_5(z)$, $\widehat{\Theta}_6(z)$, and $\widehat{\Theta}_7(z)$ on Ω that are obtained by replacing each $\boldsymbol{W}_j^{(k)}$ in $\widehat{\boldsymbol{\Theta}}_3$, $\widehat{\boldsymbol{\Theta}}_4$, $\widehat{\boldsymbol{\Theta}}_5$, $\widehat{\boldsymbol{\Theta}}_6$, and $\widehat{\boldsymbol{\Theta}}_7$ of (18.17)–(18.21) with $C_j^{(k)}(z)$ from (15.1). Next, we evaluate

```
Ce[m_,i_][z_] = W[i][z]
```

and recognize that the evaluations of `Theta[3][z]`, `Theta[4][z]`, `Theta[5][z]`, `Theta[6][z]`, `Theta[7][z]` now represent the polynomials $\widehat{\boldsymbol{\Theta}}_3$, $\widehat{\boldsymbol{\Theta}}_4$, $\widehat{\boldsymbol{\Theta}}_5$, $\widehat{\boldsymbol{\Theta}}_6$, and $\widehat{\boldsymbol{\Theta}}_7$ of (18.17)–(18.21) in the variables $\boldsymbol{W}_j^{(k)}$ over $\mathbb{Q}$.

With respect to the equations (15.9) on page 158, in order for the evaluations of `basicInv[m,1,3][z]`, `basicInv[m,1,4][z]`, `basicInv[m,1,5][z]` as well as `basicInv[m,1,6][z]` and `basicInv[m,1,7][z]` to represent the basic relative invariants $\boldsymbol{\mathcal{I}}_{m,1;\,3}$, $\boldsymbol{\mathcal{I}}_{m,1;\,4}$, $\boldsymbol{\mathcal{I}}_{m,1;\,5}$, $\boldsymbol{\mathcal{I}}_{m,1;\,6}$, and $\boldsymbol{\mathcal{I}}_{m,1;\,7}$ as polynomials over $\mathbb{Q}$ in the variables $\boldsymbol{w}_i^{(j)}$, for $1 \leq i \leq m$ and $j \geq 0$, we evaluate the input commands

```
a[m_,1][z_] := (1/Binomial[m+1,3])( w[2][z]
  -((m-1)/2)w[1]'[z]-((m-1)/(2m))w[1][z]^2 )

d[m_,1][z_] := (1/(m(m-1)))w[1][z]

K[m_,1,i_,j_][z_] := 0 /; i <= -1

K[m_,1,0,j_][z_] := 1

K[m_,1,i_,j_][z_] :=
  ( Sum[( D[K[m,1,i-1,k][z],z]
  -(m-1)*d[m,1][z]*K[m,1,i-1,k][z]
  +(m+2-i-k)(2-i-k)a[m,1][z]*
  K[m,1,i-2,k][z]),{k,j+1,m}] ) /; i >= 1

w[0][z_] = 1;    X[k_][z_] := w[k][z]

L[m_,1,i_][z_] :=
  Sum[ K[m,1,i-j,j][z]*X[j][z], {j, 0, i}]

M[m_,1,e1_,i_][z_] :=
  FunctionExpand[Binomial[m-i,e1-i]]*
  Product[(e1-r), {r,1,e1-i}]*L[m,1,i][z]

A[e1_,i_] := -1/(e1+i-1) /; i >= 1

B[e1_,i_] := (e1-i)/(e1+i-2) /; i >= 1

inv[m_,1,e1_,0][z_] := 0

inv[m_,1,e1_,1][z_] := 0

inv[m_,1,e1_,i_][z_] := ( M[m,1,e1,i][z]
  +A[e1,i-1]*D[ inv[m,1,e1,i-1][z], z]
  +B[e1,i-1]*a[m,1][z]*
  inv[m,1,e1,i-2][z] ) /; i >= 2

basicInv[m_,1,e1_][z_] := inv[m,1,e1,e1][z]
```

from pages 53–54 of Section 6.1. For the relation $\boldsymbol{w}_j^{(k)} = \binom{m}{j}\,\boldsymbol{W}_j^{(k)}$, we evaluate

```
binomial[m_,i_] := Product[m-k,{k,0,i-1}]/i!

w[i_][z_] := binomial[m,i]*W[i][z]
```

where the first of these two input commands enables $\binom{m}{i}$ to be evaluated for any nonnegative integer i even when m is merely a symbol. Since the evaluations for each of the input commands

```
FullSimplify[ basicInv[m,1,3][z] - binomial[m,3]*Theta[3][z] ]

FullSimplify[ basicInv[m,1,4][z] - binomial[m,4]*Theta[4][z] ]

FullSimplify[ basicInv[m,1,5][z] - binomial[m,5]*Theta[5][z] ]

FullSimplify[ basicInv[m,1,6][z] - binomial[m,6]*Theta[6][z] ]

FullSimplify[ basicInv[m,1,7][z] - binomial[m,7]*Theta[7][z] ]
```

is zero, we conclude that (18.24) on page 180 is valid. The *Mathematica* notebook that is downloadable from

`http://homepages.uc.edu/~chalklr/Chapter-18.html`

with the Google browser *Chrome* contains evaluations for the input statements of this chapter.

18.8. Brief summary

The subject needed a simpler notation, precise definitions, explicit formulas of a general character for the coefficients of equations resulting from a change of the independent variable, and a symmetrical development with respect to the two types of semi-invariants. Instead, after the research of Andrew Forsyth in [**28**] of 1888, the subject was identified with the performance of infinitesimal transformations. For example, see [**7**] of 1899 and [**53**] of 1906. Biographies of Georges-Henri Halphen reveal the attitudes that prevailed by incorrectly implying the subject of invariants for differential equations was merely a detail in the theory of continuous groups. Also, since Halphen's research about invariants did not fit into that context, it should have been praised rather than claimed by biographers to be "no longer in the mainstream." Thus, because the subject had become so thoroughly muddled, it needed the complete redevelopment that we began in 1989.

There are numerous areas of mathematics where considerable effort would be required for a neophyte to understand the contributions made by experts or to fit those contributions into an interesting historical perspective.

In contrast, the subject of relative invariants has a long history. Moreover, it should now be intelligible to anyone knowledgeable about the differential calculus and the concept of a polynomial in algebra.

We are pleased to have advanced this remarkable area of mathematics.

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