CHAPTER 15

Suitable Formulas for Transformations of Homogeneous Linear Differential Equations

15.1. Introduction.

For a search of the literature to find adequate formulas that yield all of the coefficients of homogeneous linear differential equations of any order m that result from changes of the independent variable, *MathSciNet* is currently not helpful. However, readers of this monograph are aware that such formulas exist because they were developed in [**14, 16, 17**] and were essential for [**19**]. The natural questions are: *how do we know that formulas like them were not previously published and why was there very little progress about relative invariants during the years* 1890–1988?

To answer these questions, library usage like that indicated in
`http://homepages.uc.edu/~chalklr/Library.pdf`
may not be possible today. However, by checking the research articles devoted to relative invariants for monic homogeneous linear differential equations, one finds that, prior to 1989, each such publication used binomial coefficients to express their equations in a form analogous to

$$y^{(m)}(z) + \sum_{i=1}^{m} \binom{m}{i} C_i(z)\, y^{(m-i)}(z) = 0, \quad \text{with } C_0(z) \equiv 1, \tag{15.1}$$

and failed to provide general transformation formulas for (15.1) corresponding to a change of the independent variable.

15.1.1. Transformations of the first kind for (15.1).

When meromorphic functions $C_1(z)$, $C_2(z)$, $\ldots$, $C_m(z)$ are given on a region Ω of the complex plane and $\rho(z)$ is a given not-identically-zero meromorphic function on Ω, there are unique meromorphic functions $C_1^*(z)$, $C_2^*(z)$, $\ldots$, $C_m^*(z)$ on Ω such that the substitution

$$y(z) = \rho(z)\, v(z), \tag{15.2}$$

viewed as a change of the dependent variable from y to v, transforms (15.1) into

$$v^{(m)}(z) + \sum_{i=1}^{m} \binom{m}{i} C_i^*(z)\, v^{(m-i)}(z) = 0, \quad \text{on } \Omega \text{ with } C_0^*(z) \equiv 1. \tag{15.3}$$

It is easy to establish directly the validity for (15.3) of

$$C_i^*(z) \equiv \sum_{j=0}^{i} \binom{i}{j} \frac{\rho^{(i-j)}(z)}{\rho(z)}\, C_j(z), \quad \text{on } \Omega \text{ when } 0 \le i \le m. \tag{15.4}$$

However, it is convenient here for us to use (15.12) and (15.13) to see immediately that (15.4) is valid. Note that $C_i^*(z)$ in (15.4) is the same for any $m \ge i$.

15.1.2. Transformations of the second kind for (15.1)**.** Let meromorphic functions $C_1(z)$, $C_2(z)$, …, $C_m(z)$ be given on a region Ω and let $z = f(\zeta)$ denote a univalent analytic function on a region Ω^{**} such that $f(\Omega^{**}) = \Omega$. Then, there are unique meromorphic functions $C_1^{**}(\zeta)$, $C_2^{**}(\zeta)$, …, $C_m^{**}(\zeta)$ on Ω^{**} such that the substitution

$$z = f(\zeta) \quad \text{with} \quad u(\zeta) = y\big(f(\zeta)\big), \tag{15.5}$$

viewed as a change of the independent variable from z to ζ, transforms (15.1) into

$$u^{(m)}(\zeta) + \sum_{i=1}^{m} \binom{m}{i} C_i^{**}(\zeta)\, u^{(m-i)}(\zeta) = 0, \quad \text{on } \Omega^{**} \text{ with } C_0^{**}(\zeta) \equiv 1. \tag{15.6}$$

However, *no previous publication has presented explicit formulas for the coefficients* $C_i^{**}(\zeta)$ *of* (15.6) *when* m *and* i *are any integers that satisfy* $1 \le i \le m$. We shall establish in Theorem 15.1 on page 159 that those coefficients are given by

$$C_i^{**}(\zeta) \equiv \sum_{j=0}^{i} \beta_{m,i-j,m-i}(\zeta)\, \big(f'(\zeta)\big)^j\, C_j\big(f(\zeta)\big), \quad \text{on } \Omega^{**} \text{ when } 0 \le i \le m, \tag{15.7}$$

where $\beta_{m,r,s}(\zeta)$ is the analytic function defined via $\alpha_{i,j}(\zeta)$ from (15.17)–(15.18) by

$$\beta_{m,r,s}(\zeta) \equiv \left[\frac{\prod_{k=1}^{r} (m-s-k+1)}{\prod_{k=1}^{r} (s+k)} \right] \alpha_{r,s}(\zeta), \quad \text{on } \Omega^{**} \text{ for } r,\, s \ge 0. \tag{15.8}$$

15.2. Consequences due to an improved notation

For given meromorphic functions $c_1(z)$, …, $c_m(z)$ on a region Ω, we write a monic mth-order homogeneous linear differential equation in the form

$$y^{(m)}(z) + \sum_{i=1}^{m} c_i(z)\, y^{(m-i)}(z) = 0, \quad \text{on } \Omega \text{ with } c_0(z) \equiv 1. \tag{15.9}$$

15.2.1. Transformations of the first kind for (15.9)**.** For any given not-identically-zero meromorphic function $\rho(z)$ on Ω, there are unique meromorphic functions $c_1^*(z)$, …, $c_m^*(z)$ on Ω such that the substitution

$$y(z) = \rho(z)\, v(z), \tag{15.10}$$

viewed as a change of the dependent variable from y to v, transforms (15.9) into

$$v^{(m)}(z) + \sum_{i=1}^{m} c_i^*(z)\, v^{(m-i)}(z) = 0, \quad \text{on } \Omega \text{ with } c_0^*(z) \equiv 1. \tag{15.11}$$

The special case $n = 1$ of (3.4) and (3.5) on page 19 yields

$$c_i^*(z) \equiv \sum_{j=0}^{i} \binom{m-j}{i-j} \frac{\rho^{(i-j)}(z)}{\rho(z)}\, c_j(z), \quad \text{on } \Omega \text{ for } 0 \le i \le m. \tag{15.12}$$

In view of $c_i(z) \equiv \binom{m}{i} C_i(z)$ and $c_i^*(z) \equiv \binom{m}{i} C_i^*(z)$, for $0 \le i \le m$, the identity

$$\binom{m-j}{i-j}\binom{m}{j}\Big/\binom{m}{i} \equiv \binom{i}{j}, \quad \text{for } 0 \le j \le i \le m, \tag{15.13}$$

shows that each of (15.4) and (15.12) is a consequence of the other.

15.2.2. Transformations of the second kind for (15.9)**.** Let $z = f(\zeta)$ denote the inverse function for a univalent analytic function $\zeta = g(z)$ on Ω. Hence, $z = f(\zeta)$ is a univalent analytic function on $\Omega^{**} = g(\Omega)$ that yields $f(\Omega^{**}) = \Omega$ and satisfies $f'(\zeta) \neq 0$, for each ζ in Ω^{**}. Then, there are unique meromorphic functions $c_1^{**}(\zeta), \ldots, c_m^{**}(\zeta)$ on Ω^{**} such that the substitution

$$z = f(\zeta) \quad \text{with} \quad u(\zeta) = y\big(f(\zeta)\big), \tag{15.14}$$

viewed as a change of the independent variable from z to ζ, transforms (15.9) into

$$u^{(m)}(\zeta) + \sum_{i=1}^{m} c_i^{**}(\zeta)\, u^{(m-i)}(\zeta) = 0, \quad \text{on } \Omega^{**} \text{ with } c_0^{**}(\zeta) \equiv 1. \tag{15.15}$$

The special case $n = 1$ of (3.21), (3.22), (3.23), and (3.24) on page 24 yields

$$c_i^{**}(\zeta) \equiv \sum_{j=0}^{i} \alpha_{i-j,m-i}(\zeta)\, \big(f'(\zeta)\big)^j\, c_j\big(f(\zeta)\big), \quad \text{on } \Omega^{**} \text{ for } 0 \leq i \leq m, \tag{15.16}$$

where the definitions of the analytic functions $\alpha_{i,j}(\zeta)$ on Ω^{**} from (3.23)–(3.24) are

$$\alpha_{0,j}(\zeta) \equiv 1, \quad \text{on } \Omega^{**} \text{ for any } j, \tag{15.17}$$

and

$$\alpha_{i,j}(\zeta) \equiv \sum_{k=1}^{j} \left[\alpha_{i-1,k}^{(1)}(\zeta) - (i+k-1)\,\frac{f''(\zeta)}{f'(\zeta)}\,\alpha_{i-1,k}(\zeta)\right], \quad \text{on } \Omega^{**} \text{ for } i \geq 1 \text{ and any } j. \tag{15.18}$$

In contrast to the development of (15.16) for (15.15) in [**19**, pages 135–137], we shall see next how the notation (15.1) complicated the situation for $C_i^{**}(\zeta)$ in (15.6).

15.3. Previously missing essential formula for older research

THEOREM 15.1. *The coefficients* $C_i^{**}(\zeta)$ *of* (15.6) *are given by* (15.7).

PROOF. For $m \geq 1$, $0 \leq i \leq m$, and ζ in Ω^{**}, we rewrite (15.16) to obtain

$$\binom{m}{i} C_i^{**}(\zeta) \equiv \sum_{j=0}^{i} \alpha_{i-j,m-i}(\zeta)\, \big(f'(\zeta)\big)^j \binom{m}{j} C_j\big(f(\zeta)\big).$$

Thus, we have

$$C_i^{**}(\zeta) \equiv \sum_{j=0}^{i} \gamma_{m,i,j}(\zeta)\, \big(f'(\zeta)\big)^j\, C_j\big(f(\zeta)\big), \tag{15.19}$$

where

$$\gamma_{m,i,j}(\zeta) \equiv \frac{\binom{m}{j}}{\binom{m}{i}}\,\alpha_{i-j,m-i}(\zeta) \equiv \left[\frac{\prod_{k=1}^{i-j}(i-k+1)}{\prod_{k=1}^{i-j}(m-i+k)}\right]\alpha_{i-j,m-i}(\zeta). \tag{15.20}$$

With $\beta_{m,r,s}$ defined by (15.8), we see that (15.20) gives $\gamma_{m,i,j}(\zeta) \equiv \beta_{m,i-j,m-i}(\zeta)$ and therefore (15.19) yields (15.7) for (15.6). This completes the proof. $\square$