CHAPTER 1

Historical Introduction

Anyone knowledgeable about the differential calculus can effortlessly develop an interest in the subject of invariants by interacting with the computer-algebra aspects of Chapters 16 and 17 before becoming involved with details.

The initial discovery of a *relative invariant* was made by Edmund Laguerre in [**37, 38**] of 1879 when he found one for monic third-order homogeneous linear differential equations. The principal research about relative invariants before 1989 was performed by Edmund Laguerre, Georges-Henri Halphen, Andrew Forsyth, and Paul Appell during the years 1879–1889. For Laguerre's use of *relative*, see page 39.

1.1. Notation to avoid

Chapters 15 and 18 provide a detailed explanation why research about relative invariants languished during the years from 1890 through 1988. The main cause was that authors who wrote papers about such matters before 1989 used notation like

$$(1.0)\quad y^{(m)}(z)+\sum_{j=1}^{m}\binom{m}{j}d_j(z)\,y^{(m-j)}(z)=0,\quad \text{with binomial coefficients } \binom{m}{j}.$$

Chapter 15 shows how that notation hindered the discovery of adequate formulas for the coefficients of equations resulting from a change of the independent variable. The abandonment of that notation in [**14, 16, 17**] during 1989–1993 was responsible for the advances in [**19, 20**]. Throughout, we shall avoid notation like that of (1.0) except for Chapters 15, 17, and 18 where early details are examined.

1.2. Relative invariant of Edmund Laguerre

To recall a result of Edmund Laguerre, we note that: when $c_1(z)$, $c_2(z)$, $c_3(z)$ are any three meromorphic functions on a region Ω of the complex plane and $\rho(z)$ is a not-identically-zero meromorphic function on Ω, there are unique meromorphic functions $c_1^*(z)$, $c_2^*(z)$, $c_3^*(z)$ on Ω such that the monic third-order homogeneous linear differential equation

$$(1.1)\qquad y'''(z)+c_1(z)\,y''(z)+c_2(z)\,y'(z)+c_3(z)\,y(z)=0,\quad \text{on } \Omega,$$

is transformed by the substitution

$$(1.2)\qquad y(z)=\rho(z)\,v(z)$$

into the monic third-order homogeneous linear differential equation

$$(1.3)\qquad v'''(z)+c_1^*(z)\,v''(z)+c_2^*(z)\,v'(z)+c_3^*(z)\,v(z)=0,\quad \text{on } \Omega.$$

Each of $c_1^*(z)$, $c_2^*(z)$, $c_3^*(z)$ can be expressed in terms of $c_1(z)$, $c_2(z)$, $c_3(z)$ and the derivatives of $\rho(z)$ by simple hand-written computations that yield

$$(*)\qquad \begin{bmatrix} c_3^*(z) - \frac{1}{3}\, c_1^*(z)\, c_2^*(z) \\ + \frac{2}{27}\left(c_1^*(z)\right)^3 - \frac{1}{2}\, c_2^{*(1)}(z) \\ + \frac{1}{3}\, c_1^*(z)\, c_1^{*(1)}(z) \\ + \frac{1}{6}\, c_1^{*(2)}(z) \end{bmatrix} \equiv \begin{bmatrix} c_3(z) - \frac{1}{3}\, c_1(z)\, c_2(z) \\ + \frac{2}{27}\left(c_1(z)\right)^3 - \frac{1}{2}\, c_2^{(1)}(z) \\ + \frac{1}{3}\, c_1(z)\, c_1^{(1)}(z) \\ + \frac{1}{6}\, c_1^{(2)}(z) \end{bmatrix}, \quad \text{for each } z \text{ in } \Omega.$$

When $c_1(z)$, $c_2(z)$, $c_3(z)$ are any three meromorphic functions defined on a region Ω and $z = f(\zeta)$ is a univalent analytic function on a region Ω^{**} with $f(\Omega^{**}) = \Omega$, there are unique meromorphic functions $c_1^{**}(\zeta)$, $c_2^{**}(\zeta)$, $c_3^{**}(\zeta)$ on Ω^{**} such that the substitution

$$(1.4)\qquad z = f(\zeta), \quad \text{with} \quad u(\zeta) = y\big(f(\zeta)\big),$$

transforms (1.1) into

$$(1.5)\qquad u'''(\zeta) + c_1^{**}(\zeta)\, u''(\zeta) + c_2^{**}(\zeta)\, u'(\zeta) + c_3^{**}(\zeta)\, u(\zeta) = 0, \quad \text{on } \Omega^{**}.$$

Each of $c_1^{**}(\zeta)$, $c_2^{**}(\zeta)$, $c_3^{**}(\zeta)$ can be expressed in terms of $c_1(z)$, $c_2(z)$, $c_3(z)$, as well as $z = f(\zeta)$ and derivatives of $f(\zeta)$ by simple computations that yield

$$(**)\qquad \begin{bmatrix} c_3^{**}(\zeta) - \frac{1}{3}\, c_1^{**}(\zeta)\, c_2^{**}(\zeta) \\ + \frac{2}{27}\left(c_1^{**}(\zeta)\right)^3 - \frac{1}{2}\, c_2^{**(1)}(\zeta) \\ + \frac{1}{3}\, c_1^{**}(\zeta)\, c_1^{**(1)}(\zeta) \\ + \frac{1}{6}\, c_1^{**(2)}(\zeta) \end{bmatrix} \equiv \big(f'(\zeta)\big)^3 \begin{bmatrix} c_3\big(f(\zeta)\big) - \frac{1}{3}\, c_1\big(f(\zeta)\big)\, c_2\big(f(\zeta)\big) \\ + \frac{2}{27}\left(c_1\big(f(\zeta)\big)\right)^3 - \frac{1}{2}\, c_2^{(1)}\big(f(\zeta)\big) \\ + \frac{1}{3}\, c_1\big(f(\zeta)\big)\, c_1^{(1)}\big(f(\zeta)\big) \\ + \frac{1}{6}\, c_1^{(2)}\big(f(\zeta)\big) \end{bmatrix}, \quad \text{for each } \zeta \text{ in } \Omega^{**}.$$

We represent the relative invariant associated with $(*)$ and $(**)$ by

$$(1.6)\qquad \boldsymbol{\mathcal{I}}_{3,1;\,3} \equiv \boldsymbol{w}_3 - \tfrac{1}{3}\boldsymbol{w}_1\boldsymbol{w}_2 + \tfrac{2}{27}(\boldsymbol{w}_1)^3 - \tfrac{1}{2}\boldsymbol{w}_2^{(1)} + \tfrac{1}{3}\boldsymbol{w}_1\boldsymbol{w}_1^{(1)} + \tfrac{1}{6}\boldsymbol{w}_1^{(2)}$$

and regard it as a differential polynomial into which substitutions can be made. Thus, with $\boldsymbol{w}_i = \boldsymbol{w}_i^{(0)}$, if $\mathcal{I}_{3,1;\,3}(z)$, $\mathcal{I}^*_{3,1;\,3}(z)$, and $\mathcal{I}^{**}_{3,1;\,3}(\zeta)$ are respectively obtained by replacing each $\boldsymbol{w}_i^{(j)}$ in $\boldsymbol{\mathcal{I}}_{3,1;\,3}$ with the corresponding $c_i^{(j)}(z)$ from (1.1), $c_i^{*(j)}(z)$ from (1.3), and $c_i^{**(j)}(\zeta)$ from (1.5), then $(*)$ and $(**)$ may be written as

$$\mathcal{I}^*_{3,1;\,3}(z) \equiv \mathcal{I}_{3,1;\,3}(z), \quad \text{on } \Omega, \text{ and } \quad \mathcal{I}^{**}_{3,1;\,3}(\zeta) \equiv \big(f'(\zeta)\big)^3\, \mathcal{I}_{3,1;\,3}\big(f(\zeta)\big), \quad \text{on } \Omega^{**}.$$

We note that the relative invariant of Edmund Laguerre presented in [**37**, page 117] of 1879, or in [**33**, page 421], corresponds to the notation $2\boldsymbol{\mathcal{I}}_{3,1;\,3}$.

Composites of some (1.2) and some (1.4) yield the transformations of (1.1).

1.3. Terminology for homogeneous linear differential equations

When $c_1(z)$, $c_2(z)$, $\ldots$, $c_m(z)$ are meromorphic functions on a region Ω of the complex plane and $\rho(z)$ is a not-identically-zero meromorphic function on Ω, there are unique meromorphic functions $c_1^*(z)$, $c_2^*(z)$, $\ldots$, $c_m^*(z)$ on Ω such that the monic mth-order homogeneous linear differential equation

$$(1.7)\qquad y^{(m)}(z) + \sum_{j=1}^{m} c_j(z)\, y^{(m-j)}(z) = 0, \quad \text{on } \Omega,$$

is transformed by the substitution

$$y(z) = \rho(z)\, v(z) \tag{1.8}$$

into the monic mth-order homogeneous linear differential equation

$$v^{(m)}(z) + \sum_{i=1}^{m} c_i^*(z)\, v^{(m-i)}(z) = 0, \quad \text{on } \Omega. \tag{1.9}$$

We set $n = 1$ in (3.4) of page 19 to obtain $c_i^*(z)$ for (1.9) as in (15.12) of page 158.

In terms of meromorphic functions $c_1(z)$, $c_2(z)$, …, $c_m(z)$ on a region Ω and a univalent analytic function $z = f(\zeta)$ on a region Ω^{**} with $f(\Omega^{**}) = \Omega$, there are unique meromorphic functions $c_1^{**}(\zeta)$, $c_2^{**}(\zeta)$, …, $c_m^{**}(\zeta)$ on Ω^{**} such that the substitution

$$z = f(\zeta), \quad \text{with} \quad u(\zeta) = y\big(f(\zeta)\big), \tag{1.10}$$

transforms (1.7) into

$$u^{(m)}(\zeta) + \sum_{i=1}^{m} c_i^{**}(\zeta)\, u^{(m-i)}(\zeta) = 0, \quad \text{on } \Omega^{**}. \tag{1.11}$$

Satisfactory formulas for the $c_i^{**}(\zeta)$ were available only after the investigations in [**14, 16, 17**] of 1989–1993 that lead to [**19**, page 136, Theorem A.3] of 2002. To verify that, see the beginning of Section 15.1 on page 157. Here, for $c_i^{**}(\zeta)$ in (1.11), set $n = 1$ in (3.21)–(3.24) on page 24 and obtain (15.16)–(15.18) on page 159.

For any polynomial $\boldsymbol{I}$ over the field $\mathbb{Q}$ of rational numbers in variables $\boldsymbol{w}_i^{(j)}$ having $1 \le i \le m$ and $j \ge 0$, let $I(z)$ on Ω, $I^*(z)$ on Ω, and $I^{**}(\zeta)$ on Ω^{**} denote the functions respectively obtained by replacing each $\boldsymbol{w}_i^{(j)}$ in $\boldsymbol{I}$ with the corresponding $c_i^{(j)}(z)$ from (1.7), $c_i^{*(j)}(z)$ from (1.9), and $c_i^{**(j)}(\zeta)$ from (1.11). Then, $\boldsymbol{I}$ is a *relative invariant* for mth-order homogeneous linear differential equations when $\boldsymbol{I}$ effectively involves at least one $\boldsymbol{w}_i^{(j)}$ and there is a fixed positive integer s such that

$$I^*(z) \equiv I(z), \quad \text{on } \Omega, \text{ and} \quad I^{**}(\zeta) \equiv \big(f'(\zeta)\big)^s I\big(f(\zeta)\big), \quad \text{on } \Omega^{**}, \tag{1.12}$$

for each equation (1.7) as well as each (1.8) and (1.10).

1.4. Relative invariants of Georges-Henri Halphen

For each fixed integer $m \ge 3$, the polynomial

$$\begin{aligned} \boldsymbol{\mathcal{I}}_{m,1;\,3} \equiv \boldsymbol{w}_3 - \frac{m-2}{m}\boldsymbol{w}_1\, \boldsymbol{w}_2 + \frac{(m-1)(m-2)}{3m^2}\big(\boldsymbol{w}_1\big)^3 - \frac{m-2}{2}\boldsymbol{w}_2^{(1)} \\ + \frac{(m-1)(m-2)}{2m}\boldsymbol{w}_1\, \boldsymbol{w}_1^{(1)} + \frac{(m-1)(m-2)}{12}\boldsymbol{w}_1^{(2)} \end{aligned} \tag{1.13}$$

is a relative invariant with $s = 3$ for the equations (1.7) of order m. This is a consequence of Theorem 4.6 on page 32. It was published in a different form by G.-H. Halphen in [**32**, page 127] of 1884. Namely, by rewriting his expression for V in [**32**, page 127, Equation (10)] or [**35**, page 112, Equation (10)] with respect to the coefficients of (1.7), we find that

$$V \equiv \left(\frac{-12}{m(m-1)(m-2)}\right) \mathcal{I}_{m,1;\,3}(z), \tag{1.14}$$

where, with $\boldsymbol{w}_i = \boldsymbol{w}_i^{(0)}$, $\mathcal{I}_{m,1;\,3}(z)$ is the function on Ω obtained by replacing each $\boldsymbol{w}_i^{(j)}$ in (1.13) with the corresponding $c_i^{(j)}(z)$ from (1.7).

In particular, for $m = 3$, (1.13) yields (1.6).

By setting $m = 4$ in (1.13), we see that the differential polynomial

$$\boldsymbol{\mathcal{I}}_{4,1;\,3} \equiv \boldsymbol{w}_3 - \tfrac{1}{2}\boldsymbol{w}_1\boldsymbol{w}_2 + \tfrac{1}{8}(\boldsymbol{w}_1)^3 - \boldsymbol{w}_2^{(1)} + \tfrac{3}{4}\boldsymbol{w}_1\boldsymbol{w}_1^{(1)} + \tfrac{1}{2}\boldsymbol{w}_1^{(2)} \tag{1.15}$$

is a relative invariant with $s = 3$ for the differential equations

$$y^{(4)}(z) + c_1(z)\,y^{(3)}(z) + c_2(z)\,y^{(2)}(z) + c_3(z)\,y^{(1)}(z) + c_4(z)\,y(z) = 0. \tag{1.16}$$

In [**31**, page 331, Equation (9)] or [**35**, page 469, Equation (9)], G.-H. Halphen had already indicated a relative invariant with $s = 3$ for the equations (1.16). His expression is equal to $(-1/2)\mathcal{I}_{4,1;\,3}(z)$. For the relative invariants presented here thus far, the computations required to directly verify their properties can be done as hand-written ones without great effort.

G.-H. Halphen used [**31**, page 339, line 3] or [**35**, page 474, line 23] of 1883 to make plausible the existence of a relative invariant with $s = 4$ for (1.16). We used Theorem 4.6 with computer algebra to conclude that the differential polynomial

$$\begin{aligned} \boldsymbol{\mathcal{I}}_{4,1;\,4} \equiv \boldsymbol{w}_4 &- \tfrac{1}{4}\boldsymbol{w}_1\boldsymbol{w}_3 - \tfrac{1}{2}\boldsymbol{w}_3^{(1)} - \tfrac{9}{100}(\boldsymbol{w}_2)^2 + \tfrac{1}{5}\boldsymbol{w}_2^{(2)} + \tfrac{13}{100}(\boldsymbol{w}_1)^2\boldsymbol{w}_2 \\ &+ \tfrac{27}{100}\boldsymbol{w}_1^{(1)}\boldsymbol{w}_2 + \tfrac{1}{4}\boldsymbol{w}_1\boldsymbol{w}_2^{(1)} - \tfrac{39}{1600}(\boldsymbol{w}_1)^4 - \tfrac{39}{200}(\boldsymbol{w}_1)^2\boldsymbol{w}_1^{(1)} \\ &- \tfrac{33}{200}\big(\boldsymbol{w}_1^{(1)}\big)^2 - \tfrac{3}{20}\boldsymbol{w}_1\boldsymbol{w}_1^{(2)} - \tfrac{1}{20}\boldsymbol{w}_1^{(3)} \end{aligned} \tag{1.17}$$

is a relative invariant with $s = 4$ for the equations (1.16). For merely a verification, see Example 16.3 on page 164. Section 12.1 shows that each relative invariant having $s = 4$ for the equations (1.16) is expressible in the form $\gamma\,\boldsymbol{\mathcal{I}}_{4,1;\,4}$, for some nonzero rational number γ.

1.5. Infinitesimal Transformations of Andrew Forsyth

To find explicit expressions for the coefficients of various relative invariants, Forsyth recognized in [**28**, pages 394–401] that computations would be considerably simplified if transformations of the type (1.10) were replaced by corresponding infinitesimal transformations where higher order infinitesimals could be neglected. His viewpoint was expressed in a footnote to [**28**, page 394] as follows.

> “The functions are shown by this process to be invariants only for an infinitesimal, but otherwise perfectly general, transformation; but the immediate purpose is to obtain the numerical coefficients and not to prove the property of general invariance, which, otherwise known, could be derived by the principle of cumulative variations.” (Andrew Forsyth)

Indeed, for each integer s satisfying $3 \le s \le 7$ and for monic homogeneous linear differential equations of order $m \ge s$, his process yields a relative invariant that satisfies (1.12) with that s. Namely, after correcting the tiny misprint that we noticed for [**28**, page 401, Equation (18)] in [**19**, page 79] and describe for $s = 7$ on page 176 of Chapter 18, we used computer algebra in [**19**] to verify that Forsyth's expressions could be identified with $\gamma_{m,s}\boldsymbol{\mathcal{I}}_{m,1;\,s}$, for $3 \le s \le 7$, where $\boldsymbol{\mathcal{I}}_{m,1;\,s}$ is given by (4.17) of page 32 with $m = m$ (as any integer $\ge s$), $n = 1$, and $e_1 = s$. Since $\boldsymbol{\mathcal{I}}_{m,1;\,s}$ was shown in [**19**] to be a relative invariant corresponding to (1.12) with $s \ge 3$ and $m \ge s$, the properties of Forsyth's expressions follow from that. Details about this are presented in Sections 18.6 and 18.7 of Chapter 18.

A direct verification for each of the formulas [**28**, pages 399–401, (14)–(18)] in the proof of Theorem 18.7 required a previously unavailable transformation formula.

1.6. Laguerre-Forsyth canonical forms

Andrew Forsyth established and applied in [**28**, pages 403–407] of 1888 the result that: for any homogeneous linear differential equation of order $m \geq 2$ having meromorphic coefficients on a region, there is a subregion on which the restriction of the equation can be transformed into a homogeneous linear differential equation of order m in which the coefficients of the derivatives of order $m-1$ and $m-2$ are zero. This process is described as a local transformation for a homogeneous linear differential equation to a Laguerre-Forsyth canonical form.

By using infinitesimal transformations with reductions to Laguerre-Forsyth canonical forms, Forsyth obtained expressions in [**28**, pages 404–407] that yield identities various relative invariants would give when restricted to transformations of one Laguerre-Forsyth canonical form into another. The corresponding invariants for this restrictive context were descriptively termed *linear invariants* by Forsyth to distinguish them from the true relative invariants of Laguerre and Halphen.

Without need for infinitesimal transformations, the preceding ideas were thoroughly redeveloped in [**19**, pages 39–49] and used in [**19**, Chapters 7–9] to prove the result [**19**, page 6, Main Theorem] that presented explicit formulas for all of the basic relative invariants of homogeneous linear differential equations. The concept of a Laguerre-Forsyth canonical form was extended in [**20**, pages 47–65, 265–274] to more general types of ordinary differential equations and their properties were essential for the verification in [**20**] of the results presented here in Theorem 4.6 on page 32 and Theorem 4.8 on page 34.

1.7. Differential equations of Paul Appell

In [**4**] of 1889, Paul Appell studied the differential equations expressible as

$$\begin{aligned}(1.18)\qquad &\big(y''(z)\big)^2 + 2\,c_{0,1}(z)\,y''(z)\,y'(z) + 2\,c_{0,2}(z)\,y''(z)\,y(z) + c_{1,1}(z)\,\big(y'(z)\big)^2 \\ &+ 2\,c_{1,2}(z)\,y'(z)\,y(z) + c_{2,2}(z)\,\big(y(z)\big)^2 = 0,\end{aligned}$$

where the $c_{i,j}(z)$ are meromorphic functions on a region Ω of the complex plane. For any not-identically-zero meromorphic function $\rho(z)$ on Ω, there are unique meromorphic functions $c^*_{i,j}(z)$ on Ω such that the substitution

$$(1.19)\qquad y(z) \equiv \rho(z)\,v(z)$$

transforms (1.18) into

$$\begin{aligned}(1.20)\qquad &\big(v''(z)\big)^2 + 2\,c^*_{0,1}(z)\,v''(z)\,v'(z) + 2\,c^*_{0,2}(z)\,v''(z)\,v(z) + c^*_{1,1}(z)\,\big(v'(z)\big)^2 \\ &+ 2\,c^*_{1,2}(z)\,v'(z)\,v(z) + c^*_{2,2}(z)\,\big(v(z)\big)^2 = 0, \quad \text{on } \Omega.\end{aligned}$$

Also, for any univalent analytic function $z = f(\zeta)$ on a region Ω^{**} with $f(\Omega^{**}) = \Omega$, there are unique meromorphic functions $c^{**}_{i,j}(\zeta)$ on Ω^{**} such that the substitution

$$(1.21)\qquad z = f(\zeta), \quad \text{with} \quad u(\zeta) = y\big(f(\zeta)\big),$$

transforms (1.18) into

$$\begin{aligned}(1.22)\qquad &\big(u''(\zeta)\big)^2 + 2\,c^{**}_{0,1}(\zeta)\,u''(\zeta)\,u'(\zeta) + 2\,c^{**}_{0,2}(\zeta)\,u''(\zeta)\,u(\zeta) + c^{**}_{1,1}(z)\,\big(u'(\zeta)\big)^2 \\ &+ 2\,c^{**}_{1,2}(\zeta)\,u'(\zeta)\,u(\zeta) + c^{**}_{2,2}(\zeta)\,\big(u(\zeta)\big)^2 = 0, \quad \text{on } \Omega^{**}.\end{aligned}$$

Simple handwritten computations show that (1.18) and (1.20) yield

$$c^{*}_{1,1}(z) - \big(c^{*}_{0,1}(z)\big)^2 \equiv c_{1,1}(z) - \big(c_{0,1}(z)\big)^2, \quad \text{on } \Omega, \tag{1.23}$$

while (1.18) and (1.22) give

$$c^{**}_{1,1}(\zeta) - \big(c^{**}_{0,1}(\zeta)\big)^2 \equiv \big(f'(\zeta)\big)^2 \left[c_{1,1}\big(f(\zeta)\big) - \Big(c_{0,1}\big(f(\zeta)\big)\Big)^2\right], \quad \text{on } \Omega^{**}. \tag{1.24}$$

In this manner, Paul Appell found the relative invariant representable by

$$\boldsymbol{\mathcal{I}}_{2,2;\,1,1} \equiv \boldsymbol{w}_{1,1} - (\boldsymbol{w}_{0,1})^2. \tag{1.25}$$

For details about other relative invariants closely related to Appell's research in [**4**], see Chapter 7. In particular, for the basic relative invariants, see page 67.

As motivation for the notation (1.29) and (1.30), we note that the differential equations (1.18) is the special case $m = 2$ of

$$\big(y^{(m)}(z)\big)^2 + \sum_{\substack{0 \le j_1,\, j_2 \le m \\ (j_1,\, j_2) \ne (0,\,0)}} c_{j_1,\, j_2}(z) \prod_{\nu=1}^{2} y^{(m-j_\nu)}(z) = 0, \tag{1.26}$$

where $c_{0,0}(z) \equiv 1$ and the coefficients $c_{j_1,j_2}(z)$ are meromorphic functions on some region Ω of the complex plane such that

$$c_{j_{\pi(1)},\, j_{\pi(2)}}(z) \equiv c_{j_1,\, j_2}(z), \quad \text{on } \Omega, \tag{1.27}$$
$$\text{for } 0 \le j_1,\, j_2 \le m \text{ and any permutation } \pi \text{ of } \{1,\, 2\}.$$

1.8. Recent developments

Earlier researchers lacked several tools essential for our work. There are now adequate transformation formulas from [**19**] and [**20**]; e.g., see Chapter 3 as well as Chapter 15. Also, computer algebra enabled us to discover several key identities for [**19, 20**]. Moreover, modern algebra provides a precise context.

From 1889 until 2002, the principal unsolved problems were the following ones.

Problem 1. For general systems of differential equations, characterize and find explicit formulas for all relative invariants that have the structure of (1.6) for the equations (1.1), the structures of (1.15) and (1.17) for the equations (1.16), the structure of (1.13) for the equations (1.7), as well as the structure of (1.25) for the equations (1.18).

Problem 2. When a satisfactory solution to Problem 1 is found and those special relative invariants are termed basic ones, discover how any relative invariant can be expressed in terms of the basic ones.

Problem 1 was solved in [**19**] and [**20**]. Namely, we characterized and found explicit formulas in [**19**] for all of the basic relative invariants for homogeneous linear differential equations. Then, after doing the same in [**20**] for nonlinear differential equations like (1.18) and (1.26), we were able to unify in [**20**, Part 4] those diverse results by characterizing and finding explicit formulas for all of the basic relative invariants of differential equations obtained as special rewritten versions of ones like

$$\big(y^{(m)}(z)\big)^n + \sum_{\substack{0 \le i_1 \le i_2 \le \cdots \le i_n \le m \\ (i_1,\, i_2,\, \ldots,\, i_n) \ne (0,\, 0,\, \ldots,\, 0)}} a_{i_1,\, i_2,\, \ldots,\, i_n}(z) \prod_{\nu=1}^{n} y^{(m-i_\nu)}(z) = 0, \quad \text{on } \Omega, \tag{1.28}$$

where m, n are positive integers and each $a_{i_1,\, i_2,\, \ldots,\, i_n}(z)$ is a meromorphic function.

Chapter 2 explains the desirability and technique for rewriting each (1.28) as

$$(1.29)\qquad \big(y^{(m)}(z)\big)^n + \sum_{\substack{0\le j_1,\,j_2,\,\dots,\,j_n\le m\\ (j_1,\,j_2,\,\dots,\,j_n)\ne(0,\,0,\,\dots,\,0)}} c_{j_1,\,j_2,\,\dots,\,j_n}(z)\prod_{\nu=1}^{n} y^{(m-j_\nu)}(z)=0,\quad \text{on } \Omega,$$

where m, n are positive integers and the $c_{j_1,\,j_2,\,\dots,\,j_n}(z)$ are meromorphic functions on some region Ω of the complex plane such that

$$(1.30)\qquad c_{0,\,0,\,\dots,\,0}(z)\equiv 1 \quad\text{and}\quad c_{j_{\pi(1)},\,j_{\pi(2)},\,\dots,\,j_{\pi(n)}}(z)\equiv c_{j_1,\,j_2,\,\dots,\,j_n}(z),$$
$$\text{for } 0\le j_1,\ j_2,\ \dots,\ j_n\le m \text{ and any permutation } \pi \text{ of } \{1,\ 2,\ \dots,\ n\}.$$

Each pair of positive integers m, n specifies a collection of equation having the form (1.29)–(1.30). For $n = 1$ and $m = m$, they are the homogeneous linear ones of (1.7) and possess relative invariants only when $m \ge 3$; while, for $m = n = 2$, they specialize to the form (1.18). Chapter 4 begins with (1.29)–(1.30) as its subject.

In terms of (1.29)–(1.30) and a not-identically-zero meromorphic function $\rho(z)$ on Ω, Theorem 3.1 specifies meromorphic functions $c^*_{m,\,n;\,j_1,\,j_2,\,\dots,\,j_n}(z)$ on Ω such that the substitution $y(z) = \rho(z)\,v(z)$ transforms (1.29)–(1.30) into

$$(1.31)\qquad \big(y^{(m)}(z)\big)^n + \sum_{\substack{0\le j_1,\,j_2,\,\dots,\,j_n\le m\\ (j_1,\,j_2,\,\dots,\,j_n)\ne(0,\,0,\,\dots,\,0)}} c^*_{m,\,n;\,j_1,\,j_2,\,\dots,\,j_n}(z)\prod_{\nu=1}^{n} y^{(m-j_\nu)}(z)=0,\quad \text{on } \Omega,$$

where

$$(1.32)\qquad c^*_{m,\,n;\,0,\,0,\,\dots,\,0}(z)\equiv 1 \quad\text{and}\quad c^*_{m,\,n;\,j_{\pi(1)},\,j_{\pi(2)},\,\dots,\,j_{\pi(n)}}(z)\equiv c^*_{m,\,n;\,j_1,\,j_2,\,\dots,\,j_n}(z),$$
$$\text{for } 0\le j_1,\ j_2,\ \dots,\ j_n\le m \text{ and any permutation } \pi \text{ of } \{1,\ 2,\ \dots,\ n\}.$$

However, the dependence of $c^*_{m,\,n;\,j_1,\,j_2,\,\dots,\,j_n}(z)$ on m and n can be implied by the context. Thus, we shall henceforth write $c^*_{j_1,\,j_2,\,\dots,\,j_n}(z)$ for $c^*_{m,\,n;\,j_1,\,j_2,\,\dots,\,j_n}(z)$. Then, (1.3), (1.9), and (1.20) are included as special cases. In particular, see (4.4).

For (1.29)–(1.30) and a univalent analytic function $z = f(\zeta)$ on a region Ω^{**} with $f(\Omega^{**}) = \Omega$, Theorem 3.3 specifies meromorphic functions $c^{**}_{m,\,n:\,j_1,\,j_2,\,\dots,\,j_n}(\zeta)$ on Ω^{**} such that $z = f(\zeta)$, with $u(\zeta) = y\big(f(\zeta)\big)$, transforms (1.29)–(1.30) into

$$(1.33)\qquad \big(u^{(m)}(\zeta)\big)^n + \sum_{\substack{0\le j_1,\,j_2,\,\dots,\,j_n\le m\\ (j_1,\,j_2,\,\dots,\,j_n)\ne(0,\,0,\,\dots,\,0)}} c^{**}_{m,\,n;\,j_1,\,j_2,\,\dots,\,j_n}(\zeta)\prod_{\nu=1}^{n} u^{(m-j_\nu)}(\zeta)=0,\quad \text{on } \Omega^{**},$$

where

$$(1.34)\qquad c^{**}_{m,\,n;\,0,\,0,\,\dots,\,0}(\zeta)\equiv 1 \quad\text{and}\quad c^{**}_{m,\,n;\,j_{\pi(1)},\,j_{\pi(2)},\,\dots,\,j_{\pi(n)}}(\zeta)\equiv c^{**}_{m,\,n;\,j_1,\,j_2,\,\dots,\,j_n}(\zeta),$$
$$\text{for } 0\le j_1,\ j_2,\ \dots,\ j_n\le m \text{ and any permutation } \pi \text{ of } \{1,\ 2,\ \dots,\ n\}.$$

The dependence of $c^{**}_{m,\,n;\,j_1,\,j_2,\,\dots,\,j_n}(\zeta)$ on m and n can be inferred from the context. Thus, we shall henceforth write $c^{**}_{j_1,\,j_2,\,\dots,\,j_n}(\zeta)$ for $c^{**}_{m,\,n;\,j_1,\,j_2,\,\dots,\,j_n}(\zeta)$. This enables (1.5), (1.11), and (1.22) to be included as special cases. Also, see (4.7).

The equations (1.29)–(1.30) possess relative invariants if and only if the fixed positive integers m and n satisfy $(m,\,n) \ne (1,\,1),\ (2,\,1)$. In regard to the situation of historical interest where $m \ge 2$, we characterize the basic relative invariants and present formulas for all of them in Sections 4.3–4.4. For the remarkably simple situation where $m = 1$ and $n \ge 2$, see [**20**, pages 257–260]. Thus, with Problem 1 solved, the emphasis of this monograph is focused on Problem 2.

1.9. Principal results not in Memoirs [**19**] and [**20**]

The subject of relative invariants for differential equations was thoroughly redeveloped in [**19**] of 2002 and [**20**] of 2007. There, the concept of a basic relative invariant was formalized and we presented a single set of explicit formulas for the construction of the basic relative invariants for a wide variety of equations. The principal problem left unsolved in [**19, 20**] was that of explicitly constructing the other relative invariants from the basic ones. At that time, methods of combining two relative invariants to construct others had not been investigated deeply.

In this revision of [**21**], we continue without alteration the examination of the construction presented on page 36 that uses relative invariants $\boldsymbol{P}$ and $\boldsymbol{Q}$ of respective weights p and q for the same type of equation to construct, for each integer $r \geq 0$, a differential-polynomial combination $\boldsymbol{C}_{p,q,r}\big(\boldsymbol{P},\, \boldsymbol{Q}\big)$ of $\boldsymbol{P}$ and $\boldsymbol{Q}$ over the field $\mathbb{Q}$ of rational numbers such that: $\boldsymbol{C}_{p,q,r}\big(\boldsymbol{P},\, \boldsymbol{Q}\big)$ is a relative invariant of weight $p+q+r$ if and only if $\boldsymbol{C}_{p,q,r}\big(\boldsymbol{P},\, \boldsymbol{Q}\big) \not\equiv 0$.

For $r \geq 2$, Theorem 4.10 of page 36 establishes that $\boldsymbol{C}_{p,q,r}\big(\boldsymbol{P},\, \boldsymbol{Q}\big)$ is a relative invariant of weight $p+q+r$ if and only if r is an even integer or $\boldsymbol{P}$ and $\boldsymbol{Q}$ are linearly independent over $\mathbb{Q}$.

For $r = 1$, we have $\boldsymbol{C}_{p,q,1}\big(\boldsymbol{P},\, \boldsymbol{Q}\big) \equiv \boldsymbol{P}\boldsymbol{Q}^{(1)} - (q/p)\,\boldsymbol{P}^{(1)}\boldsymbol{Q}$. Proposition 8.1 on page 71 shows that $\boldsymbol{C}_{p,q,1}\big(\boldsymbol{P},\, \boldsymbol{Q}\big)$ is a relative invariant of weight $p+q+1$ if and only if $\boldsymbol{P}^q$ and $\boldsymbol{Q}^p$ are linearly independent over $\mathbb{Q}$. To interpret this, see page 73.

For $r = 0$, $\boldsymbol{C}_{p,q,0}\big(\boldsymbol{P},\, \boldsymbol{Q}\big)$ is the relative invariant $\boldsymbol{P}\boldsymbol{Q}$ of weight $p+q$.

Part 1 of this monograph provides the general perspective of [**20**]. Part 2 gives a proof for Theorem 4.10. An alternate proof is given in Part 3. The results about $\boldsymbol{C}_{p,q,r}\big(\boldsymbol{P},\, \boldsymbol{Q}\big)$ are used in Part 4 with the basic relative invariants for several types of equations to construct all the relative invariants of a given weight for those equations. Part 5 clarifies the problems faced by researchers before 1989 when difficulties were created through the use of notation like that in (1.0).

1.10. Subsidiary details

To define semi-invariants of the first and second kinds for the equations (1.7), let $\boldsymbol{I}$ denote a polynomial over $\mathbb{Q}$ in variables $\boldsymbol{w}_i^{(j)}$, with $1 \leq i \leq m$ and $j \geq 0$, such that $\boldsymbol{I}$ effectively involves some $\boldsymbol{w}_i^{(j)}$. When the first condition $I^*(z) \equiv I(z)$ of (1.12) is satisfied, $\boldsymbol{I}$ is said to be a *semi-invariant of the first kind* for the equations (1.7). When the second condition $I^{**}(\zeta) \equiv \big(f'(\zeta)\big)^s I\big(f(\zeta)\big)$ of (1.12) is satisfied for some positive integer s, $\boldsymbol{I}$ is said to be a *semi-invariant of the second kind* for the equations (1.7). Thus, $\boldsymbol{I}$ is a *relative invariant* for the equations (1.7) if and only if $\boldsymbol{I}$ is both a semi-invariant of the first kind and a semi-invariant of the second kind for them. This terminology was introduced by Edmund Laguerre in [**37, 38, 33**]. For more detail about it, see Section 4.9.

The research of James Cockle in numerous papers typified by [**22**] of 1862 yielded semi-invariants of the first kind for homogeneous linear differential equations of various orders. Example 17.1 of page 168 is a reformulation of his result in [**22**]. His research in [**23**] of 1875 can be verified to give a semi-invariant of the second kind with $s = 2$ for each homogeneous linear differential equation of order $m \geq 2$. Example 17.2 of page 168 provides details. For additional information about the semi-invariantss of James Cockle, see [**7**].

As noted on page 4, Andrew Forsyth introduced infinitesimal transformations in [**28**] of 1888 to derive the coefficients for formulas thought likely to specify relative invariants of respective weights $s = 3, 4, 5, 6, 7$ for homogeneous linear differential equations of order $m \geq s$. Because his expressions illustrate well how our explicit formulas for the coefficients of transformed equations can be used to check the validity of older results, we include them as (18.17)–(18.21) on page 176. After a tiny misprint in [**28**, page 401, (18)] was corrected for (18.21), the results of Section 18.5 verify directly that (18.17)–(18.21) yield relative invariants.

Forsyth's use of infinitesimal transformations had a strong influence on later research even though his formulas corresponding to (18.17)–(18.21) were insufficient for the purpose of discovering a general pattern that would yield a relative invariant of any weight $s \geq 3$ for homogeneous linear differential equations of order $m \geq s$. In particular, Francisco Brioschi used infinitesimal transformations in [**8**, page 235] of 1891 where he presented (18.17)–(18.21) in a different form. But, his expression that corresponds to (18.21) is thoroughly incorrect. Ludwig Schlesinger employed infinitesimal transformations in [**47**, page 196] of 1897 when he included without alteration the formulas of [**8**, page 235]. Infinitesimal transformations were the focus for developments in [**7**] of 1899 and [**53**] of 1906. Moreover, they also appeared in [**26**] of 1903 for the nonlinear equations (1.18).

The most important part of the research done by Andrew Forsyth in [**28**] of 1888 for invariants of homogeneous linear differential equations was his discovery of the explicit simplified form that various relative invariants would assume when restricted to transformations of one Laguerre-Forsyth canonical form into another such form. For these expressions descriptively termed *linear invariants*, various researchers implied that they may be a key item for future progress. We have already mentioned in Section 1.6 our own indebtedness to Forsyth for those ideas.

For particular applications, various authors have proposed that a given (1.7) be locally transformed into a Laguerre-Forsyth canonical form to which Forsyth's linear invariants could then be applied. This nonconstructive procedure was suggested in [**8, 40, 47, 7, 53, 41, 42, 52, 48, 27, 43, 44**]. Because the term *relative invariant* has occasionally been incorrectly applied to situations of the preceding type that depend on a local transformation, we employed *global* as a modifier of *relative invariants* in the titles of [**19**] and [**20**]. The importance of distinguishing global properties from local ones was made clear by František Neuman in [**45**] of 1991.

While the equations (1.18) of Paul Appell were studied with respect to relative invariants in [**4, 26**] and [**20**, pages 13–17], other results about them appear in [**1, 2, 3, 4, 51, 9, 10, 12, 24, 25, 49, 13, 15, 18, 50**] and [**20**, Chapter 19]. Of unusual interest for the equations (1.18) are three relative invariants $\boldsymbol{D}_2$ in (7.2), $\boldsymbol{E}_6$ in (7.10), and $\boldsymbol{E}_7$ in (7.12) that enable us to check, as indicated on page 66, whether any given (1.18) satisfies the solvability condition (7.4) on page 65.

All three of the basic relative invariants for the equations (1.18) were initially discovered in [**20**, page 13] of 2007 by first finding in [**20**, Chapters 7–13] all of the basic relative invariants of the equations (1.26)–(1.27) for any $m \geq 2$ and then setting $m = 2$. Here, instead of regarding (1.18) as the special case of (1.26)–(1.27) having $m = 2$, we can view (1.18) as the special case of (4.1)–(4.2) having $m = 2$ and $n = 2$. Thus, we can simply set $m = 2$ and $n = 2$ in Theorem 4.6 on page 32 or in Theorem 4.8 on page 34 to obtain (7.14), (7.15), and (7.16) of page 67.

1.11. Instructive observations

MathSciNet is currently incapable of directing anyone to a publication having satisfactory formulas for all of the coefficients of the monic equation that results from a change of the independent variable in any given monic homogeneous linear differential equation of arbitrary order m. However, the satisfactory formulas of that kind developed and presented in [**19**, pages 135–137] of 2002 were essential for each of the principal advances made in [**19**], [**20**], and [**21**]. For that reason, Chapter 15 has been included as an aid for readers who are unable to use a mathematics library in the manner described in

`http://homepages.uc.edu/~chalklr/Library.pdf`

and who may therefore find it difficult to believe that there was extremely little progress about our subject from 1890 through 1988. Chapter 15 shows how the use without exception prior to 1989 of binomial coefficients as in (15.1), (15.3), and (15.6) on pages 157–158 is sufficient to explain why earlier researchers failed to discover adequate formulas for the coefficients of equations resulting from changes of the independent variable.

Chapter 16 is written as if it were a separate expository paper designed to awaken interest in a truly fascinating subject. It shows how easy it is to make interesting discoveries based on satisfactory transformation formulas and the use of computer algebra without need for additional details.

Chapter 17 demonstrates how the current existence of adequate formulas for the coefficients of transformed equations enables one to check the accuracy of results in older publications that involve the notation of (1.0) with binomial coefficients. In particular, this technique is employed in Section 18.5 to verify Theorem 18.7.

Chapter 18 introduces a suitable symbolism and precise definitions for research done before 1989 in order to show how its absence was undoubtedly a serious handicap not only to researchers but also to mathematical historians who were unable to popularize the remarkable results of 1879–1889 with precise statements.

In view of the long history of our subject, we are pleased that the single set of formulas given in Theorem 4.6 enables all of the basic relative invariants to be explicitly obtained for such a wide variety of situations. Various systems of computer algebra can easily incorporate them. Each *Mathematica* notebook in this monograph has an evaluation done with Version 7.0.1 that can be downloaded by using the Google internet browser *Chrome* to visit the web page

`http://homepages.uc.edu/~chalklr/Notebooks.htm`

and make a selection. Various other internet browsers may be unable to download these notebooks. The same evaluations for them are also produced by other versions of *Mathematica* such as Versions 8.0.1, 9.0.1, 10.1, and 11.2.