The Research about Invariants of Ordinary Differential Equations

The Research about Invariants of Ordinary Differential Equations

Roger Chalkley

Professor Emeritus of Mathematics
University of Cincinnati
Cincinnati, Ohio 45221-0025

Available from Amazon.com and other retail outlets

ISBN-10: 1985381192
ISBN-13: 978-1985381193

Chalkley, Roger 1931-
The Research about Invariants of Ordinary Differential Equations

Published by Roger Chalkley

Available from Amazon.com and other retail outlets

2010 *Mathematics Subject Classification.* 34-02, 34-04, 34A05, 34C14

Key words and phrases. relative invariant, semi-invariant, differential polynomial

Abstract. Several basic relative invariants for homogeneous linear differential equations were discovered during the years shortly after 1878. Also, a basic relative invariant was found by Paul Appell in 1889 for a type of nonlinear differential equation. There was little progress during the years 1892–1988 as researchers who worked with homogeneous linear differential equations were unknowingly handicapped by the standard practice of introducing binomial coefficients in the writing of their equations. They thereby failed to develop adequate formulas for the coefficients of equations resulting from a change of the independent variable. Consequently, for relative invariants as the most important kind of invariant, progress was stymied.

The notation was simplified in 1989, adequate transformation formulas were developed, and explicit expressions were deduced in 2002 for all of the basic relative invariants of homogeneous linear differential equations. In 2007, explicit formulas were obtained for all of the basic relative invariants of a type of ordinary differential equation involving two parameters m and n that represent positive integers. When $n = 1$ and $m \geq 3$, the formulas specialize to provide all of the basic relative invariants for homogeneous linear differential equations of order m; and, when $m = n = 2$, they yield all three of the basic relative invariants for the equations of Paul Appell.

A general method developed in 2014 combines two relative invariants of weights p and q for the same type of equation to explicitly obtain a relative invariant of weight $p + q + r$, for any $r \geq 0$. With that, the principal problems about relative invariants have now been solved.

This monograph provides clear perspective about the reformulation begun after 1988 and recently completed. Chapters 15 and 18 show how the major difficulties confronting earlier researchers have been overcome.

Contents

Preface xi

Chapter 1. Historical Introduction 1
1.1. Notation to avoid 1
1.2. Relative invariant of Edmund Laguerre 1
1.3. Terminology for homogeneous linear differential equations 2
1.4. Relative invariants of Georges-Henri Halphen 3
1.5. Infinitesimal Transformations of Andrew Forsyth 4
1.6. Laguerre-Forsyth canonical forms 5
1.7. Differential equations of Paul Appell 5
1.8. Recent developments 6
1.9. Principal results not in Memoirs [**19**] and [**20**] 8
1.10. Subsidiary details 8
1.11. Instructive observations 10

Part 1. General Perspective 11

Chapter 2. The Importance of Suitable Notation 13
2.1. The representation (2.1) for the equations of interest 13
2.2. A transitional form (2.4) for (2.1) 13
2.3. A rewritten form (2.10)–(2.11) for (2.1) that is ideal 14
2.4. Uniqueness for (2.10) subject to (2.11) 16
2.5. Representations in Chapter 4 for the context about invariants 17

Chapter 3. Coefficients of Transformed Equations 19
3.1. The coefficients $c^*_{i_1, i_2, \dots, i_n}(z)$ for (2.32)–(2.33) and (4.4)–(4.5) 19
3.2. Perspective 21
3.3. The coefficients $c^{**}_{i_1, i_2, \dots, i_n}(\zeta)$ for (2.35)–(2.36) and (4.7)–(4.8) 23
3.4. Observations about modifications versus abstractions 26
3.5. $\mathcal{C}_{m,n}$ as a set of elements 27

Chapter 4. Consistent Reformulation from 1989 Onward 29
4.1. Transformations for the equations of $\mathcal{C}_{m,n}$ 29
4.2. The ring $\mathcal{R}_{m,n}$ of differential polynomials 30
4.3. The basic relative invariants in $\mathcal{R}_{m,n}$ for $\mathcal{C}_{m,n}$ when $m \geq 2$ 32
4.4. Alternative formulas for $\boldsymbol{\mathcal{I}}_{m,n;\, e_1, \dots, e_n}$ when $m \geq 2$ 34
4.5. Principal technique for combining relative invariants 36
4.6. Several immediate applications of Theorem 4.10 37
4.7. Representations of relative invariants 38
4.8. The relative invariants in $\mathcal{R}_{3,1}$ for $\mathcal{C}_{3,1}$ of weight $s \leq 13$ 38

4.9. The terminology *relative invariant* 39
4.10. Subjects of other chapters 40

Chapter 5. Supplementary Results 41
5.1. Semi-invariants of the first kind 41
5.2. Semi-invariants of the second kind 42
5.3. The number of basic relative invariants in $\mathcal{R}_{m,n}$ for $\mathcal{C}_{m,n}$ 47
5.4. Nonsolutions of nonzero equations 48
5.5. Relative invariants in terms of basic ones and $\boldsymbol{a}_{m,n}$ 50

Chapter 6. Use of Computer Algebra 53
6.1. $\boldsymbol{\mathcal{I}}_{m,1;\,e_1}$ in $\mathcal{R}_{m,1}$ for $\mathcal{C}_{m,1}$ when $m \geq 3$ 53
6.2. Alternative computation for $\boldsymbol{\mathcal{I}}_{m,1;\,e_1}$ when $m \geq 3$ 54
6.3. $\boldsymbol{\mathcal{I}}_{m,2;\,e_1,\,e_2}$ in $\mathcal{R}_{m,2}$ for $\mathcal{C}_{m,2}$ when $m \geq 2$ 56
6.4. Alternative computation for $\boldsymbol{\mathcal{I}}_{m,2;\,e_1,\,e_2}$ when $m \geq 2$ 58
6.5. $\boldsymbol{C}_{p,q,r}\big(\boldsymbol{P},\,\boldsymbol{Q}\big)$ as a differential-polynomial combination of $\boldsymbol{P}$, $\boldsymbol{Q}$, and $\boldsymbol{a}_{m,n}$ over $\mathbb{Q}$ 59
6.6. Several identities 62
6.7. Observations about computations 63

Chapter 7. Principal Theorems Applied to Paul Appell's Study of $\mathcal{C}_{2,2}$ 65
7.1. Solution procedures for two special kinds of equations in $\mathcal{C}_{2,2}$ 65
7.2. Representations for $\boldsymbol{E}_6$, $\boldsymbol{E}_7$, and $\boldsymbol{D}_2$ 66

Chapter 8. Separate Examination of $\boldsymbol{C}_{p,q,1}\big(\boldsymbol{P},\,\boldsymbol{Q}\big)$ 71
8.1. Properties of $\boldsymbol{C}_{p,q,1}\big(\boldsymbol{P},\,\boldsymbol{Q}\big)$ 71
8.2. The condition that $\boldsymbol{P}^q$ and $\boldsymbol{Q}^p$ are linearly independent over $\mathbb{Q}$ 73
8.3. Several identities 74

Part 2. Proof of Theorem 4.10 75

Chapter 9. Invariant Character of $\boldsymbol{C}_{p,q,r}\big(\boldsymbol{P},\,\boldsymbol{Q}\big)$ when $m,\,r \geq 2$ 77
9.1. Introduction of $\boldsymbol{R}$ and $\phi_{h,i,j}(z)$ 77
9.2. Formula for $R^{**}(\zeta)$ that involves $\mathfrak{A}_{p,q,r,s,t}(\zeta)$ of (9.11) 79
9.3. Reformulation for $\mathfrak{A}_{p,q,r,s,t}\big(g(z)\big)$ 81
9.4. Initial simplification for $E_{h,i,j}(z)$ 82
9.5. Properties of $E_{h,i,j}(z)$ 87
9.6. Simplification for $\mathfrak{A}_{p,q,r,s,t}(\zeta)$ 92
9.7. $\boldsymbol{C}_{p,q,r}\big(\boldsymbol{P},\,\boldsymbol{Q}\big)$ is a relative invariant when nonzero 95

Chapter 10. Conditions for $\boldsymbol{C}_{p,q,r}\big(\boldsymbol{P},\,\boldsymbol{Q}\big) \not\equiv 0$ when $m,\,r \geq 2$ 97
10.1. The dependence of $\boldsymbol{C}_{q,p,r}\big(\boldsymbol{Q},\,\boldsymbol{P}\big)$ on $\boldsymbol{C}_{p,q,r}\big(\boldsymbol{P},\,\boldsymbol{Q}\big)$ 97
10.2. Several lemmas for use in Section 10.3 99
10.3. The situations where $\boldsymbol{C}_{p,q,r}\big(\boldsymbol{P},\,\boldsymbol{Q}\big) \not\equiv 0$ 101

Part 3. Independent Verification for Theorem 4.10 105

Chapter 11. Symmetry with Respect to Semi-Invariants 107
11.1. Context employed and definition of $\boldsymbol{S}$ 107
11.2. When $\boldsymbol{S}$ is a semi-invariant of the second kind 109

11.3. An expansion for differential polynomials like $\boldsymbol{\mathcal{P}}_k$ in (11.11) 109
11.4. Formula for $\boldsymbol{S}$ that involves $\boldsymbol{F}_{p,q,r,s,t}$ of (11.23) 111
11.5. Reformulation in (11.29) for $\boldsymbol{F}_{p,q,r,s,t}$ of (11.23) 112
11.6. Initial reformulation for $\boldsymbol{L}_{h,i,j}$ 113
11.7. Final reformulation for $\boldsymbol{L}_{h,i,j}$ 116
11.8. Simplification for $\boldsymbol{F}_{p,q,r,s,t}$ that yields $\boldsymbol{S} \equiv \boldsymbol{R}$ 121

Part 4. Relative Invariants of a Given Weight 125

Chapter 12. Representations Involving $\boldsymbol{C}_{p,q,r}\big(\boldsymbol{P},\, \boldsymbol{Q}\big)$ 127
12.1. The relative invariants in $\mathcal{R}_{4,1}$ for $\mathcal{C}_{4,1}$ of weight $s \le 12$ 127
12.2. The relative invariants in $\mathcal{R}_{5,1}$ for $\mathcal{C}_{5,1}$ of weight $s \le 12$ 128
12.3. The relative invariants in $\mathcal{R}_{2,2}$ for $\mathcal{C}_{2,2}$ of weight $s \le 12$ 130

Chapter 13. Computer Algebra for $\mathcal{V}_{3,1;\,s}$, $\mathcal{V}_{4,1;\,s}$, $\mathcal{V}_{5,1;\,s}$, $\ldots$ 133
13.1. The relative invariants of weight s in $\mathcal{R}_{m,1}$ for $\mathcal{C}_{m,1}$ 133
13.2. Simple verifications for Section 4.8 about $\mathcal{V}_{3,1;\,s}$ 139
13.3. Representation of $\boldsymbol{C}_{p,q,r}\big(\boldsymbol{P},\, \boldsymbol{Q}\big)$ in $\mathcal{R}_{m,1}$ with respect to (4.9) 139
13.4. Alternative verifications for Section 4.8 about $\mathcal{V}_{3,1;\,s}$ 140
13.5. Verifications for Section 12.1 about $\mathcal{V}_{4,1;\,s}$ 141
13.6. Verifications for Section 12.2 about $\mathcal{V}_{5,1;\,s}$ 142
13.7. Observations about versions of *Mathematica* 143

Chapter 14. Computer Algebra for $\mathcal{V}_{2,2;\,s}$, $\mathcal{V}_{3,2;\,s}$, $\mathcal{V}_{4,2;\,s}$, $\ldots$ 145
14.1. The relative invariants of weight s in $\mathcal{R}_{m,2}$ for $\mathcal{C}_{m,2}$ 145
14.2. Representation of $\boldsymbol{C}_{p,q,r}\big(\boldsymbol{P},\, \boldsymbol{Q}\big)$ in $\mathcal{R}_{m,2}$ with respect to (4.9) 151
14.3. Verifications for Section 12.3 about $\mathcal{V}_{2,2;\,s}$ 152

Part 5. Modifications Required for Developments before 1989 155

Chapter 15. Suitable Formulas for Transformations of Homogeneous Linear Differential Equations 157
15.1. Introduction. 157
15.2. Consequences due to an improved notation 158
15.3. Previously missing essential formula for older research 159

Chapter 16. Computer Algebra with Formulas (15.9)–(15.18) 161
16.1. Computer representations for $c_i^*(z)$ and $c_i^{**}(\zeta)$ 161
16.2. Applications based on the representations for $c_i^*(z)$ and $c_i^{**}(\zeta)$ 162

Chapter 17. Computer Algebra with Formulas (15.1)–(15.8) 167
17.1. Computer-algebra representations of $C_i^*(z)$ and $C_i^{**}(\zeta)$. 167
17.2. Computer-algebra verifications. 168

Chapter 18. Suitable Context for Older Notation 171
18.1. Symbolism and terminology 171
18.2. Our viewpoint abut the older Cockle-semi-invariants 172
18.3. Original introduction of $\widehat{G}_i(z)$, $\widehat{G}_i^*(z)$, and $\widehat{G}_i^*(z) \equiv \widehat{G}_i(z)$ 175
18.4. Results of Forsyth in the context for Sections 18.1 and 18.2 176
18.5. Computer-algebra verification of (18.23) 178
18.6. Several observations 180

18.7. Computer-algebra verification of (18.24) 180
18.8. Brief summary 182

Bibliography 183

Index 187

Preface

The subject of relative invariants for ordinary differential equations has been completely redeveloped in a series of publications begun in 1989. Now, there are satisfactory solutions to the principal unsolved problems that provided interest for researchers after Edmund Laguerre found a relative invariant in [**37, 38**] of 1879 for third-order homogeneous linear differential equations and the French Academy of Sciences encouraged extensions of his research. In particular, Georges-Henri Halphen won the 1880 Grand Prize of the French Academy of Sciences for research about invariants published in [**32**] and Henri Poincaré received honorable mention for his competitive submission to them in 1880.

Explicit formulas for all of the basic relative invariants of homogeneous linear differential equations of each fixed order were found and presented in [**19**] of 2002. For a type of nonlinear differential equation studied by Paul Appell in [**4**] of 1889, he discovered one of its three basic relative invariants. The other two were obtained for [**20**] of 2007 and all three appear in [**20**, page 13, Theorem 1.8] of 2007.

As a remarkable generalization not anticipated by earlier researchers, all of the basic relative invariants were discovered and presented by explicit formulas in [**20**, pages 257, 264, 275–276] for a type of ordinary differential equation involving two integral parameters m and n, where m is the order of the equation and n is its degree when its left member is regarded as a homogeneous polynomial in the various derivatives of the dependent variable. In particular, when $n = 1$, the formulas specialize to yield the ones in [**19**] for the basic relative invariants of homogeneous linear differential equations of each order $m \geq 3$; and, when $m = n = 2$, they specialize to yield the ones in [**20**] for the three basic relative invariants of the nonlinear equations Paul Appell studied in [**4**].

To complete the research involving the preceding results, a construction was developed in [**21**] of 2014 where, under general conditions, it combines two relative invariants of respective weights p and q for the same type of equation to produce a relative invariant of weight $p + q + r$, for any integer $r \geq 0$. Examples were also given in [**21**] to illustrate how, starting with the basic relative invariants for a given type of equation, that construction can be repeatedly applied to obtain linearly independent relative invariants of a given weight whose linear combinations yield all of the relative invariants having that weight.

This revision of [**21**] includes Chapters 15 and 18 as new ones to show why, after a flurry of intense interest during the years 1879–1891, the subject remained in limbo until 1989. In particular, these chapters make precise the principal difficulties earlier researchers failed to overcome.

Roger Chalkley