An Interesting Question Whose Answer Has Been Overlooked

Roger Chalkley, December 26, 2014

Here, we provide a clear answer to the following question.

> Although each of the highly-respected mathematicians Laguerre, Halphen, Forsyth, and Appell made interesting discoveries about basic relative invariants for differential equations during the years 1879–1889, the main problem of discovering explicit formulas for all of the basic relative invariants was nowhere near a solution during the years from 1879 through 1988. What happened to enable one person to discover explicit formulas for all of the basic relative invariants during the years 1989–2007?

The answer is that: from earliest times until 1989, when any monic homogeneous linear differential equation $L = 0$ of arbitrary order m with independent variable z is transformed by a univalent change $z = f(\zeta)$ of the independent variable from z to ζ in order to obtain a corresponding unique monic homogeneous linear differential equation $L^{**} = 0$ whose coefficients and dependent variable are functions of ζ, no adequate formulas had been developed to explicitly express the coefficients of $L^{**} = 0$ in terms of the coefficients of $L = 0$ as well as m and $z = f(\zeta)$. This means that, if one bothers to search the literature as described here, one would recognize that the various implied formulas presented for that purpose before 1989 can not be implemented with m as a mere symbol for an integer. In contrast, satisfactory transformation formulas corresponding to a change of the dependent variable were available from early times onward.

For details about the first and only satisfactory transformation formulas corresponding to changes of the independent variable, see my 2002 and 2007 Memoirs. Here is a question.

> What advice should a librarian X at a mathematical-research library Y give a mathematical researcher Z when Z is serious about finding for his research the best general formulas currently available for transforming homogeneous linear differential equations under a change of the independent variable?

For that particular question, I know that no librarian or anyone else can cite any particular review in any volume of *Mathematical Reviews* or *Zentralblatt für Mathematik und ihre Grenzgebiete* or their on-line successors that would be useful in answering that question. The best procedure that one could currently use for that purpose consists of three steps. (1) Use the reference journals to find every publication about homogeneous linear differential equations likely to be non-superficial and to involve transformations. (2) Carefully read acquired copies of each of the possibly helpful publications and eliminate each one that does not attempt to provide information about the coefficients of $L^{**} = 0$ for any given $L = 0$ of arbitrary order m under a change $z = f(\zeta)$ of the independent variable. (3) For each of the likely few publications remaining, use a computer-algebra system to see if the publication enables a computer to be programmed so that each coefficients of $L^{**} = 0$ is expressed explicitly in terms of the coefficients of $L = 0$, the arbitrary order m of $L = 0$, and $z = f(\zeta)$.

If all of the paper copies at library Y of *Mathematical Reviews* and the *Zentralblatt für Mathematik und ihre Grenzgebiete* are away in storage, perhaps X should simply say to Z something like: instead of answering that question, why not do something practical such as become a plumber, or an electrician, or join other mathematicians investigating newly created abstractions where history is ignorable and mutual praise is profuse. In a few years time, Wikipedia may be able to help for the subject of this page.