Plan of talk

**Markov processes with Nonhomogeneous transition probabilities**

- Contraction coefficient. Dobrushin central limit theorem for Markov chains.


- Nonstationary martingale approximation.
Dobrushin’s contraction coefficient

Dobrushin’s contraction coefficient as a measure of degeneracy

Let $Q$ be a Markov transition probability on $(X, \mathcal{B}(X))$

$$Q(x, A) = P(\xi_1 \in A | \xi_0 = x)$$

$$(Qu)(x) = \int_X u(y) Q(x, dy).$$

Dobrushin’s contraction coefficient

$$\delta(Q) = \sup_{x_1, x_2, A} |Q(x_1, A) - Q(x_2, A)|$$

$$\delta(Q) = \sup_{u \in \mathcal{U}} \sup_{x_1, x_2} |(Qu)(x_1) - (Qu)(x_2)|,$$

where $\mathcal{U} = \{u, \text{Osc}(u) = \sup_{x_1, x_2} |u(x_1) - u(x_2)| \leq 1\}.$

Dobrushin’s coefficient of independence

$$\alpha(Q) = 1 - \delta(Q)$$
Properties of Dobrushin’s coefficients

\[ 0 \leq \delta(Q) \leq 1 \]

\[ \delta(Q) = 0 \text{ if and only if } Q(x, A) \text{ does not depend on } x \]

\( Q \) is called nondegenerate if \( 0 \leq \delta(Q) < 1 \)

\[ \delta(Q_1 Q_2) \leq \delta(Q_1) \delta(Q_2) \]

For a matrix \( Q = (q_{i,j}) \)

\[ \alpha(Q) = 1 - \delta(Q) = \inf_{i,j \in I} \sum_{k \in I} \min(q_{i,k}, q_{j,k}) \]
Non-homogeneous transitions

Markov chain with nonstationary transitions.

- \((\xi_i)_{1 \leq i \leq n}\) non-homogeneous Markov chain of length \(n\) with values in \((\mathcal{X}, \mathcal{B}(\mathcal{X}))\)

\[
P(\xi_{i+1} \in A | \xi_i = x) = Q_{i,i+1}(x, A)
\]

\[
Q_{i,j} = Q_{i,i+1} Q_{i+1,i+2} \cdots Q_{j-1,j}
\]

\[
\delta_1 = \sup_{1 \leq i \leq n-1} \delta(Q_{i,i+1}) \quad \text{and} \quad \delta_k = \sup_{1 \leq i \leq n-k} \delta(Q_{i,i+k}).
\]

\[
\delta_k \leq \delta_1
\]
Let \((\xi_{n,i})_{1 \leq i \leq n}\) be an array of non-homogeneous Markov chains with values in \((\mathcal{X}, \mathcal{B}(\mathcal{X}))\) and

\[ X_{n,i} = f_{n,i}(\xi_{n,i}) \text{ and } S_n = \sum_{i=1}^{n} X_{n,i} , \]

\[ \mathbb{E}X_{n,i} = 0, \quad \mathbb{E}X_{n,i}^2 < \infty \]

and denote by

\[ \sigma_n^2 = \text{var } S_n \text{ and } b_n^2 = \sum_{i=1}^{n} \text{var } X_{n,i} , \]

\[ \delta_{n,1}, \quad \alpha_n = 1 - \delta_{n,1} . \]
Dobrushin’s CLT

**Theorem**

Under the assumptions

\[
\max_{1 \leq i \leq n} |X_{n,i}| \leq C_n \text{ a.s. and } \frac{C_n^2}{\alpha_n^3 b_n^2} \to 0 \text{ as } n \to \infty ,
\]

we have

\[
\frac{\sum_{i=1}^n X_{n,i}}{\sigma_n} \xrightarrow{D} N(0, 1) \text{ as } n \to \infty .
\]

**Corollary**

When \(|X_{n,i}| \leq C < \infty \) a.s. and \(\text{var } X_{n,i} \geq c > 0\), then the CLT holds provided

\[
\alpha_n^3 n \to \infty \text{ as } n \to \infty .
\]
There is a Markov chain with $|X_{n,i}| \leq C < \infty$ a.s. and $\text{var} X_{n,i} \geq c > 0$, and $\limsup \alpha^3 n < \infty$ which does not satisfy the CLT.

Example $0 < p < 1$

$$Q = \begin{pmatrix} 1 - p & p \\ p & 1 - p \end{pmatrix}, \quad \pi(1) = \pi(2) = \frac{1}{2}, \quad \alpha(Q) = 2p$$

$$T_n = \sum_{i=1}^{n} 1\{1\}(X_i)$$

- For $p = 1/n$, $T_n/n \Rightarrow G$ and $\lim_{n \to \infty} n^2 \text{var}(T_n) = V_1 < \infty$
- For $p = 1 - 1/n$, $T_n - n/2 \Rightarrow F$ and $\lim_{n \to \infty} \text{var}(T_n) = V_2 < \infty$
Blocking argument: $\alpha_n \to 0$, $n^{1/3} \alpha_n \geq 1$ and make about $n\alpha_n$ blocks of size $\alpha_n^{-1}$, $l_1, l_2, \ldots, l_{n\alpha_n}$.

Define:

$$Q_{i,i+1} = Q(\alpha_n) \text{ for } 1 \leq i \leq \alpha_n^{-1}$$

$$= Q(1/2) \text{ for } i = j\alpha_n^{-1}, \ 1 \leq j \leq n\alpha_n$$

$$= Q(1 - \alpha_n) \text{ in rest}$$

$$T_n = \sum_j \sum_{i \in l_j} 1\{1\}(X_i) = \sum_j T_{n,j}$$

$$\text{var}T_{n,1} \sim \alpha_n^{-2} V_1 \quad \text{var}\left(\sum_{j \geq 2} T_{n,j}\right) \sim n\alpha_n V_2$$

If $n^{1/3} \alpha_n \to \infty$ then CLT. If $n^{1/3} \alpha_n = 1$ the limiting distribution is a convolution $X \ast N$, where $X$ is non-normal nondegenerate.
Maximal coefficient of correlation

Let $\mathcal{A}, \mathcal{B}$ be two sub $\sigma$-algebras

$$\rho(\mathcal{A}, \mathcal{B}) = \sup_{f \in L_2(\mathcal{A}), g \in L_2(\mathcal{B})} |\text{corr}(f, g)|,$$

For a vector of random variables $(Y_k)_{1 \leq k \leq n}$ we define

$$\rho_k = \max_{1 \leq s, s+k \leq n} \rho(\sigma(Y_i, i \leq s), \sigma(Y_j, j \geq s + k)).$$

For a nonhomogeneous Markov chain of length $n$, $(\xi_i)_{1 \leq i \leq n}$,

$$\rho_k = \max_{1 \leq s, s+k \leq n} \rho(\sigma(\xi_s), \sigma(\xi_{s+k})).$$

if the joint distribution of $(\xi_s, \xi_{s+k})$ is bivariate normal

$$\rho_k = \max_{1 \leq s, s+k \leq n} |\text{corr}(\xi_s, \xi_{s+k})|.$$
Relation to Dobrushin’s coefficient

(Bradley 2011) For any $0 < a < b < 1$ there is a Markov chain with $\rho_1 = a$ and $\delta_1 = b$. Therefore, for a triangular array, $\lambda_n = 1 - \rho_{n,1}$ and $\alpha_n = 1 - \delta_{n,1}$ can converge to 0 at different rates.

Sethuraman and Varadhan (2005)

$$\rho_1 \leq \delta_1^{1/2}. \quad (*)$$

For a Markov chain

$$\rho_k \leq \rho_1^k. \quad (\text{**})$$
Relation to operator norm

Relation with operator norm in $L_2$

$$\rho_1(X, Y) = \sup_g \left\{ \frac{\| E(g(Y) | X) \|_2^2}{\| g(Y) \|_2^2} ; \| g(Y) \|_2 < \infty \text{ and } Eg(Y) = 0 \right\},$$

For stationary Markov chains with values in $(X, \mathcal{B}(X))$ and transition probability $Q(x, A) = P(\xi_1 \in A | \xi_0 = x)$, define the operator $Q$

$$(Qu)(x) = \int_X u(y) Q(x, dy), u \in L_2(X, \mathcal{B}(X), \pi)$$

Denote $L^0_2(\pi) = \{ g \in L_2(X, \mathcal{B}(X), \pi) \text{ with } \int g d\pi = 0 \}$. The coefficient $\rho_1$ is simply the norm operator of $Q : L^0_2(\pi) \rightarrow L^0_2(\pi)$,

$$\rho_1 = \| Q \|_{L^0_2(\pi)} = \sup_{g \in L^0_2(\pi)} \frac{\| Q(g) \|_2^2}{\| g \|_2^2}.$$

(Institute)  

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Some history on the maximal coefficient of correlation

Conditions imposed to the maximal coefficient of correlation make possible to study the asymptotic behavior of many dependent structures including classes of Markov chains and Gaussian sequences. This coefficient was used by Kolmogorov and Rozanov and further studied by Rosenblatt, Ibragimov, Shao among many others. An introduction to this topic, mostly in the stationary setting, can be found in the Chapters 9 and 11 in Bradley. Application to the central limit theorem (CLT) for various stationary Markov chains with $\rho_1 < 1$ are surveyed in Jones. In the nonstationary setting and general triangular arrays a central limit theorem was obtained by Utev, assuming $\rho$—mixing coefficients converging to 0 uniformly at a logarithmic rate.
CLT under conditions on the maximal coefficient of correlation

**Theorem**

(P-2010) Suppose that \((X_{n,i})_{1 \leq i \leq n}\) are Markov chains and

\[
\max_{1 \leq i \leq n} |X_{n,i}| \leq C_n \text{ a.s. and } \frac{C_n(1 + |\log(\lambda_n)|)}{\lambda_n^{3/2} b_n} \to 0 \text{ as } n \to \infty.
\]

Then

\[
\frac{\sum_{i=1}^{n} X_{n,i}}{\sigma_n} \overset{D}{\to} N(0, 1) \text{ as } n \to \infty.
\]

**Corollary**

Assume that \(\max_{1 \leq i \leq n} |X_{n,i}| \leq C \text{ a.s. and also } \text{var } X_{n,i} \geq c > 0 \text{ for all } n \geq 1 \text{ and } 1 \leq i \leq n. \) Then CLT holds provided

\[
\lambda_n^3 n(1 + |\log(\lambda_n)|)^{-2} \to \infty.
\]
Integral form

\[
\frac{1}{\lambda_n^2 b_n^2} \sum_{i=1}^{n} \mathbb{E} X_{n,i}^2 I(|X_{n,i}| > \varepsilon h(\lambda_n) b_n) \rightarrow 0 \text{ as } n \rightarrow \infty
\]

where \( h(\lambda_n) = \lambda_n^{3/2} (1 + |\log(\lambda_n)|)^{-1} \).

**Uniformly bounded \( \rho_{n,1} \) mixing coefficients.**

There is a positive number \( \rho \) such that \( \sup_n \rho_{n,1} < \rho < 1 \). Then, the CLT holds provided that for every \( \varepsilon > 0 \)

\[
\frac{1}{b_n^2} \sum_{i=1}^{n} \mathbb{E} X_{n,i}^2 I(|X_{n,i}| > \varepsilon b_n) \rightarrow 0.
\]
Assume \((X_{n,j})_{1 \leq j \leq n}\) is an array of centered random variables that are square integrable and adapted to an increasing array of sigma fields \((\mathcal{F}_{n,j})_{1 \leq j \leq n}\).

**Theorem**

Assume \(E S_n^2 = 1\). Define the projector operator

\[
P_{n,j} Y = E(Y|\mathcal{F}_{n,j}) - E(Y|\mathcal{F}_{n,j-1}).
\]

Assume

\[
\max_{1 \leq j \leq n} |P_{n,j} S_n| \to^P 0 \text{ as } n \to \infty.
\]

and

\[
\sum_{j=1}^{n} (P_{n,j} S_n)^2 \to^P 1 \text{ as } n \to \infty.
\]

Then the CLT holds.
For $0 \leq j \leq n$ denote

$$A_{n,j} = \mathbb{E}(S_n - S_{n,j} \mid \mathcal{F}_{n,j}) ,$$

where $S_{n,j} = \sum_{i=1}^{j} X_{n,i}$

**Theorem**

Assume $\mathbb{E}S_n^2 = 1$. Also assume that

$$\max_{1 \leq j \leq n} (|X_{n,j}| + |A_{n,j}|) \xrightarrow{P} 0 \text{ as } n \to \infty$$

and

$$\sum_{j=1}^{n} (X_{n,j}^2 + 2X_{n,j}A_{n,j}) \xrightarrow{P} 1 \text{ as } n \to \infty .$$

Then $S_n$ converges in distribution to $N(0,1)$. 
Bounds for the variance of partial sums of Markov chains

Sharp upper and lower bounds for the variance of partial sums of a Markov chain in function of the maximal coefficient of correlation

\[
\frac{1 - \rho_1}{1 + \rho_1} \sum_{i=1}^{n} EX_i^2 \leq ES_n^2 \leq \frac{1 + \rho_1}{1 - \rho_1} \sum_{i=1}^{n} EX_i^2 .
\]

and in function of Dobrushin’s coefficient

\[
\frac{1 - \delta_1}{(1 + \sqrt{\delta_1})^2} \sum_{i=1}^{n} EX_i^2 \leq ES_n^2 \leq \frac{(1 + \sqrt{\delta_1})^2}{1 - \delta_1} \sum_{i=1}^{n} EX_i^2 .
\]
Quenched Invariance principle for Stationary processes and Markov chains

Markov processes with homogeneous transitions that do not start from equilibrium.

- Quenched Central Limit Theorem for stochastic processes
- Quenched Central Limit theorem for Reversible Markov processes
- Stationary Martingale Approximation
Stationary Sequences

Stationary sequence as a functional of a Markov chain with a general state space.

- \((\xi_n)_{n \in \mathbb{Z}}\) is a stationary ergodic Markov chain defined on a probability space \((\Omega, \mathcal{F}, \mathbb{P})\) with values in a measurable space.
- The marginal distribution is denoted by \(\pi(A) = \mathbb{P}(\xi_0 \in A)\).
- \(S_n = S_n(g) = \sum_{k=1}^{n} g(\xi_k)\)
- Any stationary sequence \((Y_k)_{k \in \mathbb{Z}}\) can be viewed as a function of a Markov process \(\xi_k = (Y_j; j \leq k)\) with the function \(g(\xi_k) = Y_k\).
For any continuous and bounded $f$

\[ \mathbb{E}^x(f(S_n/\sqrt{n})) \to E(f(cN)) \text{, a.s.} \]

where $N$ is standard normal. Equivalently,

\[ \mathbb{E}(f(S_n/\sqrt{n})|\mathcal{F}_0) \to E(f(cN)) \text{, a.s.} \]

For $t \in [0, 1]$ define $W_n(t) = S_{[nt]}/\sqrt{n}$, where $[x]$ denotes the integer part of $x$.

By quenched functional CLT we mean that for any function $f$ continuous and bounded on $D(0, 1)$

\[ \mathbb{E}(f(W_n)|\mathcal{F}_0) \to E(cf(W)) \text{ a.s.} \]

where $W$ is the standard Brownian motion on $[0, 1]$. 
Almost sure martingale approximations

Martingales with stationary and ergodic differences have both almost sure CLT and invariance principle started at a point. Type of approximations needed to transport the Quenched CLT from martingale to the stationary sequence.

\[ \mathbb{E}(\frac{(S_n - M_n)^2}{n} | F_0) / n \to 0 \text{ a.s.} \]

\[ \mathbb{E}(\max_{1 \leq i \leq n} (S_i - M_i)^2 | F_0) / n \to 0 \text{ a.s.} \]

\[ \frac{(S_n - M_n)^2}{n} \to 0 \text{ a.s.} \]
Here is a short history of the quenched CLT under projective criteria.

- A result in Borodin and Ibragimov (1994) states that if $\|\mathbb{E}(S_n|\mathcal{F}_0)\|_2$ is bounded, then the CLT in its functional form started at a point holds.
- Later work by Derriennic and Lin (2001) improved on this result imposing the condition $\|\mathbb{E}(S_n|\mathcal{F}_0)\|_2 = O(n^{-1/2-\varepsilon})$ with $\varepsilon > 0$.
- Improvements by Wu and Woodroofe (2004); Zhao and Woodroofe (2008).
- A step forward was made by Cuny (2009) who improved the condition to $\|\mathbb{E}(S_n|\mathcal{F}_0)\|_2 = O(n^{-1/2}(\log n)^2(\log \log n)^{1+\delta})$ by using sharp results on ergodic transforms in Gaposhkin (1996).
- Peligrad and Cuny (2011) under the condition

$$ \sum_{k \geq 1} \frac{\|\mathbb{E}(X_k|\mathcal{F}_0)\|_2}{\sqrt{k}} = \sum_{k \geq 1} \frac{\|Q^k f\|_2}{\sqrt{k}} < \infty . $$
Additive functionals of reversible Markov chains

Additive functionals of reversible Markov chains.

**Theorem**

Let \((\xi_i)_{i \in \mathbb{Z}}\) be a stationary and ergodic reversible Markov chain, 
\(f \in L^2_0(\pi)\) with the property

\[
\lim_{n \to \infty} \frac{\text{var}(S_n)}{n} \to \sigma_f^2 < \infty \quad \text{(equivalently} \quad \sum_n \frac{||\mathbb{E}(S_n | \mathcal{F}_0)||^2_2}{n^2} < \infty \). 

Then \(\mathbb{E} \max_{1 \leq j \leq n} (S_j - M_j)^2 / n \to 0\) and the functional CLT holds.


\(-Q = Q^*\) where \(Q^*\) is the adjoint operator defined by

\[
\langle Qf, g \rangle = \langle f, Q^*g \rangle, \text{ for every } f \text{ and } g \text{ in } L^2_2(\pi). 
\]
Conjecture: Kipnis and Varadhan. For any square integrable function of stationary, ergodic, reversible Markov chains under centering and normalization $\sqrt{n}$, the functional central limit theorem started from a point holds under the condition $\lim_{n \to \infty} \frac{\text{var}(S_n)}{n} \to \sigma_f^2 < \infty$ (equivalently $\sum_n \frac{\|E(S_n|\mathcal{F}_0)\|_2^2}{n^2} < \infty$). The same question for continuous time reversible Markov chains.
Quenched CLT for Reversible Markov chains

Cuny Peligrad (2011).

**Theorem**

For a stationary reversible Markov chain satisfying

\[ \sum_n \frac{\left(\log \log n\right)^2 \left\| \mathbb{E}(S_n | \mathcal{F}_0) \right\|_2^2}{n^2} < \infty. \]

the central limit theorem started at a point holds.
Method of proof:

Construction of approximating martingale
- $\mathbb{E}(S_n|\mathcal{F}_1) - \mathbb{E}(S_n|\mathcal{F}_0) \to D_0$ in $L_2$ and a.s.
- Then one estimates $S_n - \sum_{k=1}^{n} D_k$ in $L_2$
- Almost sure-type martingale approximation based on maximal inequalities
- Spectral calculus
Fourier series

Application of martingale approximation to Fourier Series

\[ S_n(\theta) = \sum_{j=1}^{n} X_j \exp(ji\theta) \text{ where } i^2 = -1, \]

periodogram \( I_n(\theta) = 2\pi n^{-1} |S_n(\theta)|^2 \)

Wiener and Wintner (1941), Lacey and Terwilleger (2008):
For all stationary sequences \((X_j)_{j \in \mathbb{Z}}\) in \(\mathbb{L}_1\) there is a set \(\Omega'\) of probability 1 such that

\[ \frac{S_n(\theta)}{n} \text{ converges for all } \theta \text{ and } \omega \in \Omega' \]
Theorem

\((X_k)_{k \in \mathbb{Z}} \) is a stationary ergodic sequence and \( \mathbb{E}(X_0 | \mathcal{F}_{-\infty}) = 0 \) almost surely. Then for \( \lambda \)-almost all \( \theta \in (0, 2\pi) \)

\[
\lim_{n \to \infty} \mathbb{E}|S_n(\theta)|^2 / n = \sigma^2(\theta)
\]

and

\[
\frac{1}{\sqrt{n}} (\text{Re}(S_n(\theta)), \text{Im}(S_n(\theta))) \to^d (N_1(\theta), N_2(\theta))
\]

where \( N_1(\theta) \) and \( N_2(\theta) \) are independent identically distributed normal random variables mean 0 and variance \( \frac{1}{2} \sigma^2(\theta) \).

MP-Wei Biao Wu (2009)

(*) Regularity can be replaced by \( \|\mathbb{E}_0(S_n(\theta))\|_2 = o(\sqrt{n}) \).
Method of proof of central limit theorem for Fourier series

Carleson and Hunt theorems imply for almost all $\theta$ in $[0, 2\pi]$

$\mathbb{E}_1(S_n(\theta)) - \mathbb{E}_0(S_n(\theta)) \rightarrow D_0$ in $L_2$

Martingale approximation.
Spectral density theory.