SEMINAR TALK: WICK FORMULA FOR QUATERNION NORMAL LAWS

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Abstract. This is a seminar talk of February 13, 2008.

1. Wick's theorem for $\mathbb{R}$-valued normal vectors

The following is known as Wick's theorem.

Theorem A (Wick [9])**.** *If $(X_1, \dots, X_{2n})$ is multivariate normal with mean zero, then*

$$E(X_1X_2\dots X_{2n}) = \sum_V \prod_{\{j,k\}\in V} E(X_jX_k),$$

where the sum is taken over all pair partitions V of $\{1, 2, \dots, 2n\}$.[1]

For example, $E(X_1X_2X_3X_4) = E(X_1X_2)E(X_3X_4)+E(X_1X_3)E(X_2X_4)+E(X_1X_4)E(X_2X_3)$ as $\{1, 2, 3, 4\}$ has three pair partitions $V_1 = \{\{1, 2\}, \{3, 4\}\}$, $V_1 = \{\{1, 3\}, \{2, 4\}\}$, $V_1 = \{\{1, 4\}, \{3, 2\}\}$. In particular, if $X_1 = X_2 = X_3 = X_4 = X$ and $E(X^2) = 1$, then the formula gives $E(X^4) = 3$.

Proof. (This is a consequence of moments-cumulants relation [4]; the connection is best visible in the partition formulation of [7]. For another proof, see [2, page 12].) Suppose first that $X_1 \dots X_n$ are of very special form. Namely, suppose they are selected with repetition from a fixed finite i.i.d. $N(0, 1)$ family $Z_1, Z_2, \dots$. Then $E(X_1 \dots X_n)$ is zero if some of the $Z's$ enter the product an odd number of times. If all Z's appear an even number of times, then the answer is $E(X_1 \dots X_n) = (2n_1-1)!!(2n_2-1)!! \dots (2n_k-1)$, where say Z_1 is repeated $2n1$ times, Z_2 is repeated $2n_2$ times, etc.

This matches the answer from the pair partitions: the only contributing partitions are those that pair $\{1, \dots, 2n_1\}$ within itself, and $\{n_1 + 1, \dots, n_1 + n_2\}$ within itself, etc. The number of pairings of $\{1, \dots, 2n\}$ is $(2n - 1)$ matches for 1 times the number of pairings of the remaining $2n - 2$ elements. Thus it is $(2n - 1) \times (2n - 3) \times \dots 3 \times 1$.

Suppose now that we have general multivariate normal $X_j = \sum_j A_{i,j}Z_j$. Then $C_{a,b} = E(X_aX_b) = \sum A_{a,j}A_{b,j}$.

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[1] That is, partitions into two-element sets, so each V has the form

$$V = \{\{j_1, k_1\}, \{j_2, k_2\}, \dots, \{j_n, k_n\}\}.$$

Now by the previous part,

$$E(X_1,\dots,X_{2n}) = \prod_i \sum_j A_{ij}Z_j = \sum_{j_1,\dots,j_{2n}} A_{1j_1}\dots A_{2n,j_{2n}} E(Z_{j_1}\dots Z_{j_{2n}})$$

$$= \sum_{j_1,\dots,j_{2n}} A_{1j_1}\dots A_{2n,j_{2n}} \sum_V \prod_{(a,b)\in V} \delta_{j_a=j_b} = \sum_{j_1,\dots,j_{2n}} \sum_V \prod_{(a,b)\in V} \delta_{j_a=j_b} A_{a,j_a}A_{b,j_b}$$

$$= \sum_V \prod_{(a,b)\in V} \left(\sum_{j_a,j_b} \delta_{j_a=j_b} A_{a,j_a}A_{b,j_b} \right) = \sum_V \prod_{(a,b)\in V} \sum_{j_a} A_{a,j_a}A_{b,j_a}$$

$$= \sum_V \prod_{(a,b)\in V} C_{a,b} = \sum_V \prod_{(a,b)\in V} E(X_a,X_b).$$

□

Example 1.1. The above proof works also for $X_1,\dots,X_n$ that are linear combinations of i.i.d. $N(0,1)$ r.v. with complex coefficients A_{ij}. So with $Z = X+iY$, where X,Y are independent $N(0,1)$ random variables, we get $E(Z^{2n}) = 0$ and $E(|Z|^{2n}) = n!$.

2. Wick's Theorem for quaternion normal variables

2.1. **Quaternions.** Recall that a quaternion $q\in\mathbb{H}$ can be represented as $q = x_0+ix_1+jx_2+kx_3$ with $i^2=j^2=k^2=ijk=-1$ and real coefficients $x_0,\dots,x_3$. The conjugate quaternion is $\overline{q} = x_0-ix_1-jx_2-kx_3$, so $|q|^2 := q\overline{q}\geq 0$. Quaternions with $x_1=x_2=x_3$ are usually identified with real numbers; the real part of a quaternion is $\Re(q) = (q+\bar{q})/2$.

It is well known that quaternions can be identified with the set of certain 2×2 complex matrices:

$$\mathbb{H}\ni x_0+ix_1+jx_2+kx_3 \sim \begin{bmatrix} x_0+ix_1 & x_2+ix_3 \\ -x_2+ix_3 & x_0-ix_1 \end{bmatrix} \in \mathcal{M}_{2\times 2}(\mathbb{C}) \tag{2.1}$$

where on the right hand side i is the usual imaginary unit of $\mathbb{C}$. Note that since $\Re(q)$ is twice the trace of the matrix representation in (2.1), this implies cyclic property $\Re(q_1q_2) = \Re(q_2q_1)$ which we will use freely.

2.2. **Quaternion normal law.** The (standard) quaternion normal random variable is a $\mathbb{H}$-valued random variable which can be represented as

$$Z = \xi_0 + i\xi_1 + j\xi_2 + k\xi_3 \tag{2.2}$$

with independent real normal $N(0,1)$ random variables ξ_0,ξ_1,ξ_2,ξ_3. Due to symmetry of the centered normal laws on $\mathbb{R}$, the law of $(Z,\overline{Z})$ is the same as the law of $(\overline{Z},Z)$. A calculation shows that if Z is quaternion normal then for fixed $q_1,q_2\in\mathbb{H}$,

$$\mathbb{E}(Zq_1Zq_2) = \mathbb{E}(Z^2)\bar{q}_1q_2,\ \mathbb{E}(Zq_1\overline{Z}q_2) = \mathbb{E}(Z\bar{Z})\Re(q_1)q_2\,.$$

For future reference, we insert explicitly the moments:

$$\mathbb{E}(Zq_1Zq_2) = -2\bar{q}_1q_2, \tag{2.3}$$

$$\mathbb{E}(Zq_1\overline{Z}q_2) = 2(q_1+\bar{q}_1)q_2. \tag{2.4}$$

By linearity, these formulas imply

$$\mathbb{E}(\Re(Zq_1)\Re(\overline{Z}q_2)) = \Re(q_1q_2), \tag{2.5}$$

$$\mathbb{E}(\Re(Zq_1)\Re(Zq_2)) = \Re(\bar{q}_1q_2). \tag{2.6}$$

2.3. **Wick's formula.** In view of the Wick formula for real-valued jointly Gaussian random variables, formulas (2.3) and (2.4) allow us to compute moments of certain products of quaternion normal random variables. Suppose the n-tuple $(X_1, X_2, \dots, X_n)$ consists of random variables taken, possibly with repetition, from the set $\{Z_1, \bar{Z}_1, Z_2, \bar{Z}_2, \dots\}$ where $Z_1, Z_2, \dots$ are independent quaternion normal. Consider an auxiliary family of independent pairs $\{(Y_j^{(r)}, Y_k^{(r)}) : r = 1, 2, \dots\}$ which have the same laws as (X_j, X_k), $1 \leq j, k \leq n$ and are independent for different r. Then the Wick formula for real-valued normal variables implies $\mathbb{E}(X_1, X_2, \dots, X_n) = 0$ for odd n, and

$$\mathbb{E}(X_1X_2\dots X_n) = \sum_f \mathbb{E}(Y_1^{(f(1))}Y_2^{(f(2))}\dots Y_n^{(f(n))}), \tag{2.7}$$

where the sum is over all two-to-one valued functions $f : \{1, \dots, n\} \to \{1, \dots, m\}$ for $n = 2m$. Formulas (2.3) and (2.4) then show that the Wick reduction step takes the following form.

$$\mathbb{E}(X_1X_2\dots X_n) = \sum_{j=2}^{n} \mathbb{E}(X_1X_j)\mathbb{E}(U_jX_{j+1}\dots X_n) \tag{2.8}$$

where

$$U_j = \begin{cases} \Re(X_2\dots X_{j-1}) & \text{if } X_j = \bar{X}_1 \\ \bar{X}_{j-1}\dots\bar{X}_2 & \text{if } X_j = X_1 \\ 0 & \text{otherwise} \end{cases}$$

For example, if Z is quaternion normal then applying (2.7) with the pairings $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$ we get

$$\mathbb{E}(Z^4) = \mathbb{E}(X^2)(\mathbb{E}(Z^2) + \mathbb{E}(\bar{Z}Z) + \mathbb{E}(\bar{Z}^2)) = 0.$$

This suggests that one can do inductively the calculations, but it is not quite clear what answers to expect. We begin with the one-variable warm-up.

Proposition 2.1. *If Z is quaternion normal* (2.2)*, then:*

$$\mathbb{E}(|Z|^{2k}) = 2^k(k+1)! \tag{2.9}$$

$$\mathbb{E}(Z^2|Z|^{2k-2}) = -2^{k-1}(k+1)! \tag{2.10}$$

$$\mathbb{E}(Z^{2m}|Z|^{2k}) = 0,\ m > 1. \tag{2.11}$$

Proof of (2.9), (2.10), (2.11). Clearly, formula (2.9) holds true when $k = 0, 1$. Formula (2.10) holds for $k = 0$. Formula (2.11) holds for $k = 0$, $m = 2$ by inspection.

Suppose there is a $K \geq 0$ such that the formulas already hold for ALL k, m such that $k + m \leq K$. Then the recurrence step is based on the expansions:

$$E(Z^2|Z|^{2K}) = E(ZZ^{K+1}\bar{Z}^K) \tag{2.12}$$

$$E(|Z|^{2K+2}) = E(ZZ^K\bar{Z}^{K+1}) \tag{2.13}$$

$$E(Z^{2+2m}|Z|^{2k}) = E(ZZ^{k+2m+1}\bar{Z}^k) \tag{2.14}$$

(2.15)

Each of the inductive steps applies (2.9), (2.10), (2.11) to the expressions with the total exponent of at most K.

Using (2.8), we expand the right hand side of (2.12):

$$E(Z^2|Z|^{2K}) = E(ZZ^{K+1}\bar{Z}^K) = \sum_{j=1}^{K+1} E(Z^2)E(\bar{Z}^{K+j-1}Z^{K+1-j}) \tag{2.16}$$
$$+\frac{1}{2}\sum_{r=1}^{K} E(|Z|^2)E(Z^{K+1}\bar{Z}^{K-1}) + \frac{1}{2}\sum_{r=1}^{K} E(|Z|^2)E(\bar{Z}^{2K+1-r}Z^{r-1})$$

Substituting $u = K+2-j$ in the first sum, we get

(2.17)

$$E(Z^2|Z|^{2K}) = -2\sum_{u=1}^{K+1} E(\bar{Z}^{2u-2}|Z|^{2K+2-2u}) + 2KE(Z^2|Z|^{2K-2}) + 2\sum_{r=1}^{K} E(\bar{Z}^{2K+2-2r}|Z|^{2r-2})$$
$$= -2E(|Z|^{2K}) - 2E(\bar{Z}^2|Z|^{2K-2}) - 2K2^{K-1}(K+1)! + 2E(Z^2|Z|^{2K-2})$$
$$= -2\times 2^K(K+1)! - 2K2^{K-1}(K+1)! = -2^K(K+1)!(K+2)$$

Note that the proof used (2.11) with pairs k, m such that $m > 1$ and $k+m \le K$.

Using (2.8), we expand (2.13). (This just requires us to shift the index K by 1 in the previous proof.)

$$E(|Z|^{2K+2}) = E(ZZ^K\bar{Z}^{K+1}) = \sum_{j=1}^{K} E(Z^2)E(\bar{Z}^{K+j}Z^{K-j})$$
$$+\frac{1}{2}\sum_{r=1}^{K+1} E(|Z|^2)E(Z^K\bar{Z}^K) + \frac{1}{2}\sum_{r=1}^{K+1} E(|Z|^2)E(\bar{Z}^{2K+1-r}Z^{r-1})$$
$$= -2E(\bar{Z}^{K+1}Z^{K-1}) + 2(K+1)E(Z^K\bar{Z}^K) + 2E(\bar{Z}^KZ^K) + 2E(\bar{Z}^{K+1}Z^{K-1})$$
$$= -2E(Z^2|Z|^{2K-2}) + 2(K+1)E(|Z|^{2K}) + 2E(|Z|^{2K}) + 2E(Z^2|Z|^{2K-2})$$
$$= 2(K+2)E(|Z|^{2K}) = 2(K+2)2^K(K+1)! = 2^{K+1}(K+2)!$$

Note that the proof used (2.11) for pairs k, m such that $m > 1$ and $k+m \le K$.

Suppose now $m > 1$ is such that $k+m = K$. Using (**??**), we expand (2.14):

(2.18)

$$E(Z^{2+2m}|Z|^{2k}) = E(ZZ^{k+2m+1}\bar{Z}^k) = \sum_{j=1}^{k+2m+1} E(Z^2)E(\bar{Z}^{k+j-1}Z^{k+2m+1-j})$$
$$+\frac{1}{2}\sum_{r=1}^{k} E(|Z|^2)E(Z^{k+2m+1}\bar{Z}^{k-1}) + \frac{1}{2}\sum_{r=1}^{k} E(|Z|^2)E(\bar{Z}^{2k+2m+1-r}Z^{r-1})$$
$$= -2\sum_{j=1}^{k+2m+1} E(\bar{Z}^{2j-2}|Z|^{2k+2m+2-2j}) + 2kE(Z^{2m+2}|Z|^{2k-2}) + 2\sum_{r=1}^{k} E(\bar{Z}^{2k+2m+2-2r}|Z|^{2r-2})$$
$$= -2E(|Z|^{2k+2m}) - 2E(Z^2|Z|^{2k+2m-2}) + 0$$
$$= -2\times 2^{k+m}(k+m+1)! + 4\times 2^{k+m-1}(k+m+1)! = 0.$$

Note that again the proof uses (2.11) only with $k+m \leq K$. Similarly, (2.9) and (2.10) are applied only to exponents $\leq K$.

$\square$

Formula (2.8) implies that $\mathbb{E}(X_1X_2\dots X_n)$ is real, so on the left hand side of (2.8) we can write $\mathbb{E}(\Re(X_1X_2\dots X_n))$; this form of the formula is associated with one-vertex Möbius graphs.

Furthermore, we have a Wick reduction in multiple vertex cases

$$\mathbb{E}(\Re(X_1)\Re(X_2X_3\dots X_n)) = \sum_{j=2}^{n} \mathbb{E}(\Re(X_1)\Re(X_j))\mathbb{E}(\Re(X_2\dots X_{j-1}X_{j+1}\dots X_n)), \tag{2.19}$$

(this is just the consequence of Wick formula for the $\mathbb{R}$-valued case). Furthermore, from (2.1) it is clear that we have cyclic symmetry,

$$\Re(X_1X_2\dots X_n) = \Re(X_2X_3\dots X_nX_1). \tag{2.20}$$

3. Möbius graphs and quaternion normal moments

3.1. **Möbius graphs.** Möbius graphs are graphs (with loops) that allow regular as well as "twisted" edges. For the formal definition, see [6, Section 6]. Here we will draw the vertices as disks, the edges as ribbons which may preserve orientation, or reverse it (twists).

3.1.1. *Euler characteristics.* Denote by $v(\Gamma)$, $e(\Gamma)$, and $f(\Gamma)$ the number of vertices, edges, and faces of Γ. The Euler characteristic is

$$\chi(\Gamma) = v(\Gamma) - e(\Gamma) + f(\Gamma).$$

Note that the faces are the cycles of the graph. To illustrate the orientation-changing edges, we draw ribbons for edges, and disks for the vertices. The convention is that the cycle is not completed until we return to the "same side" of the disk that represents a vertex. For example, in Fig. 1, the Euler characteristics are $\chi_1 = 1-1+2 = 2$ and $\chi_2 = 1-1+1 = 1$.

If Γ decomposes into components Γ_1, Γ_2, then $\chi(\Gamma) = \chi(\Gamma_1) + \chi(\Gamma_2)$.

3.2. **Quaternion version of Wick's theorem.** Suppose the n-tuple $(X_1, X_2, \dots, X_{2n})$ consists of random variables taken, possibly with repetition, from the set $\{Z_1, \bar{Z}_1, Z_2, \bar{Z}_2, \dots\}$ where $Z_1, Z_2, \dots$ are independent quaternion normal. Fix a sequence $j_1, j_2, \dots, j_m$ of natural numbers such that $j_1 + \dots + j_m = 2n$.

Consider the family $M = M_{j_1,\dots,j_m}(X_1, X_2, \dots, X_{2n})$, possibly empty, of Möbius graphs with m vertices of degrees $j_1, j_2, \dots, j_m$ which are labeled by $X_1, X_2, \dots, X_{2n}$, whose regular edges correspond to pairs $X_i = \bar{X}_j$ and flipped edges correspond to pairs $X_i = X_j$. No edges of $\Gamma \in M$ can join random variables X_i, X_j that are independent.

Theorem 3.1.

$$\begin{aligned}\mathbb{E}\big(\Re(X_1X_2\dots X_{j_1})\Re(X_{j_1+1}\dots X_{j_1+j_2}) \times \dots \\ \times \Re(X_{j_1+j_2+j_{m-1}+1}\dots X_{2n})\big) = 4^{n-m}\sum_{\Gamma\in M}(-2)^{\chi(\Gamma)}.\end{aligned} \tag{3.1}$$

(The right hand side is interpreted as 0 when $M = \emptyset$.)

Question 3.2. What would be the formula for $E(q_1X_1q_2X_2\dots q_{2n}X_{2n})$? Or for more general Q-Gaussian n-touples? (For more general definition, see [8])

Proof. In view of (2.7) and (2.8), it suffices to show that if $X_1,\dots,X_{2n}$ consists of n independent pairs, and each pair is either of the form (X,X) or $(X,\bar X)$, then

$$\mathbb{E}\big(\Re(X_1X_2\dots X_{j_1})\Re(X_{j_1+1}\dots X_{j_1+j_2})\times\dots \\ \times\Re(X_{j_1+j_2+j_{m-1}+1}\dots X_{2n})\big)=4^{n-m}(-2)^{\chi(\Gamma)}, \tag{3.2}$$

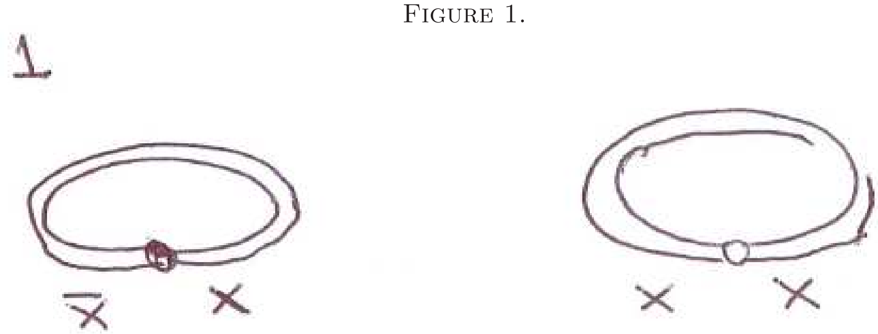
where Γ is the Möbius graph that describes all pairings of the sequence.

First we check the two Möbius graphs for $n=1$, $m=1$:

$$\mathbb{E}(\Re(X\bar X))=(-2)^2\,,\quad\text{and}\quad \mathbb{E}(\Re(XX))=(-2)^1\,.$$

One checks that these correspond to the Möbius graphs in Figure 1, which gives a sphere ($\chi=2$) and projective sphere ($\chi=1$) respectively.

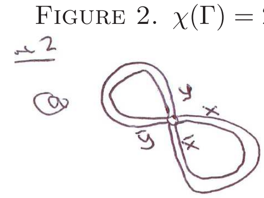
Figure 1.



For $n=2,m=1$ we have the following twelve cases.

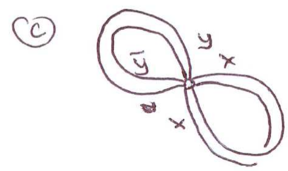
(1)
$$\mathbb{E}(\Re(XY\bar Y\bar X))=\mathbb{E}(X\bar X)\mathbb{E}(Y\bar Y)=(-2)^2 4^1$$
which matches the graph in Figure 2. This graph gives a sphere.

Figure 2. $\chi(\Gamma)=2$



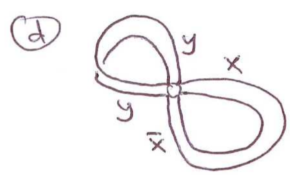
(2)
$$\mathbb{E}(\Re(XY\bar YX))=\mathbb{E}(XX)\mathbb{E}(Y\bar Y)=(-2)^1 4^1$$
which matches the graph in Figure 3. This graph gives a projective sphere.

FIGURE 3. $\chi(\Gamma) = 1$



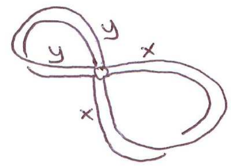
(3)
$$\mathbb{E}(\Re(XYY\bar{X})) = \mathbb{E}(X\bar{X})\mathbb{E}(\Re(YY)) = (-2)^1 4^1$$
which matches the graph in Figure 4. This graph gives a projective sphere.

FIGURE 4. $\chi(\Gamma) = 1$



(4)
$$\mathbb{E}(\Re(XYYX)) = \mathbb{E}(XX)\mathbb{E}(\bar{Y}\bar{Y}) = 4^1$$
which matches the graph in Figure 5. This graph gives a Klein bottle.

FIGURE 5. $\chi(\Gamma) = 0$

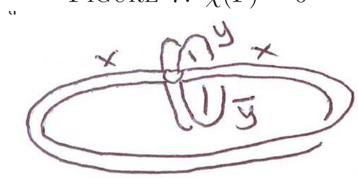


(5)
$$\mathbb{E}(\Re(XY\bar{X}\bar{Y})) = \mathbb{E}(X\bar{X})\mathbb{E}(\Re(Y)\bar{Y}) = 4^1$$
which matches the graph in Figure 6. This graph gives a torus.

FIGURE 6. $\chi(\Gamma) = 0$

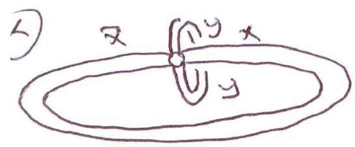
(6)
$$\mathbb{E}(\Re(XYX\bar{Y})) = \mathbb{E}(XX)\mathbb{E}(\bar{Y}\bar{Y}) = 4^1$$
which matches the graph in Figure 7. This graph gives a Klein bottle.

FIGURE 7. $\chi(\Gamma) = 0$



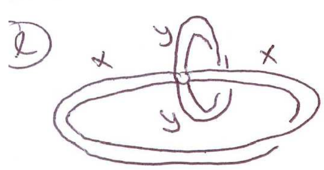
(7)
$$\mathbb{E}(\Re(XY\bar{X}Y)) = \mathbb{E}(X\bar{X})\mathbb{E}(\Re YY) = 4^1$$
which matches the graph in Figure 8. This graph gives a Klein bottle.

FIGURE 8. $\chi(\Gamma) = 0$



(8)
$$\mathbb{E}(\Re(XYXY)) = \mathbb{E}(XX)\mathbb{E}(\bar{Y}Y) = (-2)^1 4^1$$
which matches the graph in Figure 9. This graph gives a projective sphere.

FIGURE 9. $\chi(\Gamma) = 1$



(9)
$$\mathbb{E}(\Re(X\bar{X}Y\bar{Y})) = \mathbb{E}(X\bar{X})\mathbb{E}(Y\bar{Y}) = (-2)^2 4^1$$
which matches the graph in Figure 10. This graph gives a sphere.

(10)
$$\mathbb{E}(\Re(XXY\bar{Y})) = \mathbb{E}(XX)\mathbb{E}(Y\bar{Y}) = (-2)^1 4^1$$
which matches the graph in Figure 11. This graph gives a projective sphere.

FIGURE 10. $\chi(\Gamma) = 2$

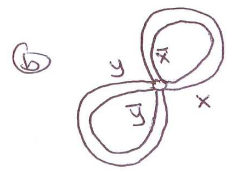
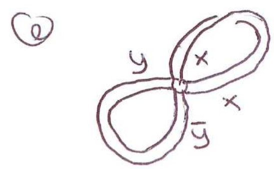
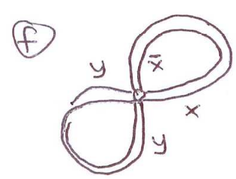


FIGURE 11. $\chi(\Gamma) = 1$



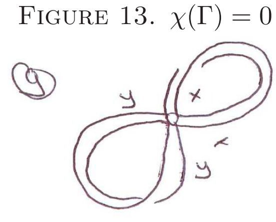
(11)
$$\mathbb{E}(\Re(X\bar{X}YY)) = \mathbb{E}(X\bar{X})\mathbb{E}(YY) = (-2)^1 4^1$$
which matches the graph in Figure 12. This graph gives a projective sphere.

FIGURE 12. $\chi(\Gamma) = 1$



(12)
$$\mathbb{E}(\Re(XXYY)) = \mathbb{E}(XX)\mathbb{E}(YY) = 4^1\,,$$
which matches the graph in Figure 13. This graph gives a Klein bottle.

FIGURE 13. $\chi(\Gamma) = 0$



We leave the cases of $n = 2$, $m = 2$, to the reader, they proceed as above using only the relations (2.3-2.6).

(1) $\mathbb{E}(\Re(XX)\Re(YY) = 4$, $\chi = 2$.

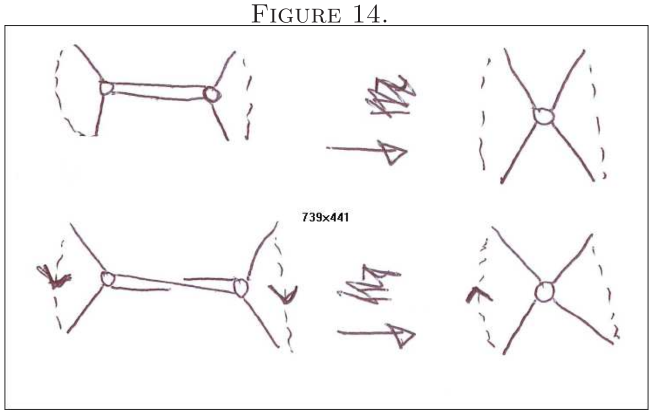
(2) $\mathbb{E}(\Re(XX)\Re(Y\bar{Y}) = -8$, $\chi = 3$.
(3) $\mathbb{E}(\Re(X\bar{X})\Re(Y\bar{Y}) = 16$, $\chi = 4$.
(4) $\mathbb{E}(\Re(XY)\Re(X\bar{Y}) = -2$, $\chi = 1$.
(5) $\mathbb{E}(\Re(XY)\Re(XY) = 4$, $\chi = 2$.
(6) $\mathbb{E}(\Re(X)\Re(\bar{X}Y\bar{Y}) = \mathbb{E}(\Re(X)\Re(XY\bar{Y}) = \mathbb{E}(\Re(X)\Re(Y\bar{X}\bar{Y}) = \mathbb{E}(\Re(X)\Re(YX\bar{Y}) = 4$, $\chi = 2$.
(7) $\mathbb{E}(\Re(X)\Re(XYY) = \mathbb{E}(\Re(X)\Re(YXY) = \mathbb{E}(\Re(X)\Re(Y\bar{X}Y) = -2$, $\chi = 1$

We now proceed with the induction step. One notes that by independence of the pairs at different edges, the left hand side of (3.2) factors into the product corresponding to connected components of Γ. It is therefore enough to consider connected Γ.

If Γ has two vertices that are joined by an edge, we can use cyclicity of $\Re$ to move the variables that label the edge to the first positions in their cycles, say X_1 and X_{j_1} and use (2.5) or (2.6) to eliminate this pair from the product. The use of relation (2.5) is just that of gluing the two vertices together removing the edge x which is labeled by the two appearances of Z. Relation (2.6) glues together the two vertices, removing the edge x, and the reversal of orientation across the edge is given by the conjugate (see Figure 14). These geometric operations reduce n and m by one without changing the Euler characteristic: the number of edges and the number of vertices are reduced by 1; the faces are preserved – in the case of edge flip in Fig. 14, the edges of the face from which we remove the edge, after reduction follow the same order.

Therefore we will only need to prove the result for the single vertex case of the induction step.

Figure 14.



We wish to show that

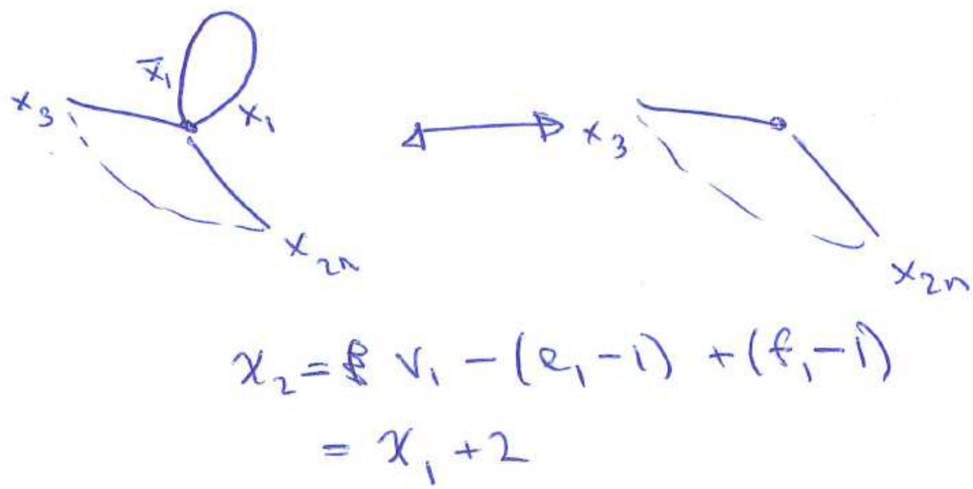
$$\mathbb{E}(X_1X_2X_3X_4\dots X_{2n}) = (-2)^{\chi(\Gamma)}4^{n-1}\,, \tag{3.3}$$

where Γ is a one vertex Möbius graph with arrows (half edges) labeled by X_k. The trivial case is when $X_1 = \bar{X}_2$ in which case we find that (3.3) is

$$\begin{aligned}\mathbb{E}(X_1\bar{X}_1)\mathbb{E}(X_3X_4\dots X_{2n}) &= 4\mathbb{E}(X_3X_4\dots X_{2n}) \\ &= 4\left[(-2)^{\chi_2}4^{(n-1)-1}\right] \\ &= (-2)^{\chi_2}4^{n-1}\,.\end{aligned} \tag{3.4}$$

Equation (3.4) corresponds to the reduction of the Möbius graph given in Figure 15 and represents a net of 0 to the Euler characteristic, $\chi_2 = \chi_1 + 2$, from which we see that the result follows in this case.

FIGURE 15. $\chi_2 = v_2 - e_2 + f_2 = v_1 - (e_1 - 1) + (f_1 - 1) = \chi_1$



The second nearly trivial case is $X_2 = X_1$ in which case we find that (3.3) is

$$\mathbb{E}(X_1X_1)\mathbb{E}(X_3X_4\ldots X_{2n}) = (-2)\left[\mathbb{E}(X_3X_4\ldots X_{2n})\right] \tag{3.5}$$

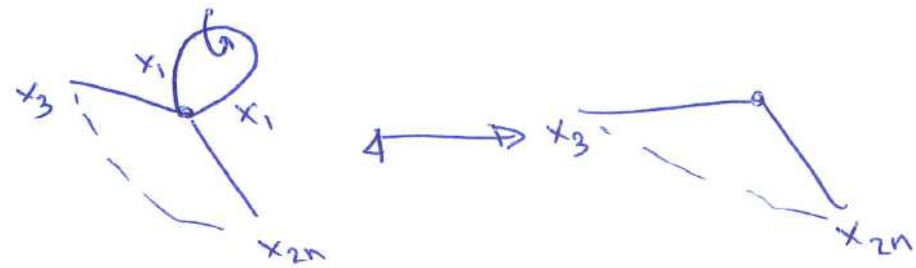
$$= (-2)\left[(-2)^{\chi_2}4^{(n-1)-1}\right] \tag{3.6}$$

$$= (-2)^{\chi_2 - 1}4^{n-1}\,. \tag{3.7}$$

Equation (3.5) corresponds to the reduction of the Möbius graph given in Figure 16 and represents a net change of 1 to the Euler characteristic, from which we see that the result follows in this case.

In what follows we take $j < k$, and let $q_1 = X_3 \ldots X_{j-1}$, $q_2 = X_{j+1} \ldots X_{k-1}$, and $q_3 = X_{k+1} \ldots X_{2n}$. Conjugation of q_i in the following computations arises from an orientation reversal. In the diagrams we have used an arrow to indicate an edge which reverses the local orientation of the vertex, in other words a ribbon with a flip to it. Geometrically conjugation reverses the order of the edges as well as switching local orientations. There are 8 non-trivial cases, corresponding precisely to the $n = 2$, $m = 1$ cases 1-8:

FIGURE 16.

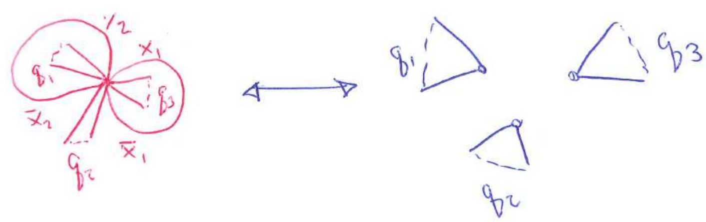


$$\chi_2 = v_1 - (e_1 - 1) + f_1$$
$$= \chi_1 + 1$$

(1) When $X_1 = \bar{X}_k$ and $X_2 = \bar{X}_j$ we find that

$$\begin{aligned}
&\mathbb{E}(X_1 X_2 q_1 \bar{X}_2 q_2 \bar{X}_1 q_3) \\
&= \mathbb{E}(X_1 \bar{X}_1) \mathbb{E}(\Re(X_2 q_1 \bar{X}_2 q_2) \Re(q_3)) \\
&= 4\mathbb{E}(X_2 \bar{X}_2) \mathbb{E}(\Re(q_1) \Re(q_2) \Re(q_3)) \\
&= 4^2 \mathbb{E}(\Re(q_1) \Re(q_2) \Re(q_3)) \\
&= 4^2 \left[(-2)^{\chi_2} 4^{(n-2)-3} \right] \\
&= (-2)^{\chi_2 - 4} 4^{n-1} .
\end{aligned}$$

FIGURE 17.



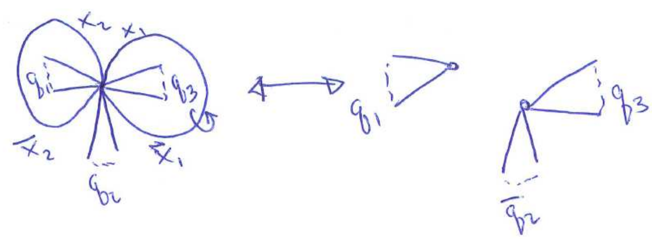
$$\chi_2 = (e_1 + 2) - (e_1 - 2) + f_1$$
$$= \chi_1 + 4$$

This corresponds to the reduction of the Möbius graph given in Figure 17. We note that $\chi_2 = v_2 - e_2 + f_2 = v_1 + 2 - (e_1 - 2) + f_1 = \chi_1 + 4$, as the number of faces is not changed when $q_1, q_2, q_3 \neq 1$.

(2) When $X_1 = X_k$ and $X_2 = \bar{X}_j$ we find that

$$\begin{aligned}
&\Re\mathbb{E}(X_1 X_2 q_1 \bar{X}_2 q_2 X_1 q_3)\\
&= \mathbb{E}(X_1 X_1)\Re\mathbb{E}(\bar{q}_2 X_2 \bar{q}_1 \bar{X}_2 q_3)\\
&= (-2)\Re\mathbb{E}(X_2 \bar{q}_1 \bar{X}_2 q_3 \bar{q}_2)\\
&= (-2)\mathbb{E}(X_2 \bar{X}_2)\mathbb{E}(\Re(\bar{q}_1)\Re(q_3 \bar{q}_2))\\
&= (-2)^3\mathbb{E}(\Re(\bar{q}_1)\Re(q_3 \bar{q}_2))\\
&= (-2)^3\left[(-2)^{\chi_2} 4^{(n-2)-2}\right]\\
&= (-2)^{\chi_2 - 3} 4^{n-1}\,.
\end{aligned}$$

FIGURE 18.



$$\begin{aligned}
\chi_2 &= (v_1 + 1) - (e_1 - 2) + f_1\\
&= \chi_1 + 3
\end{aligned}$$

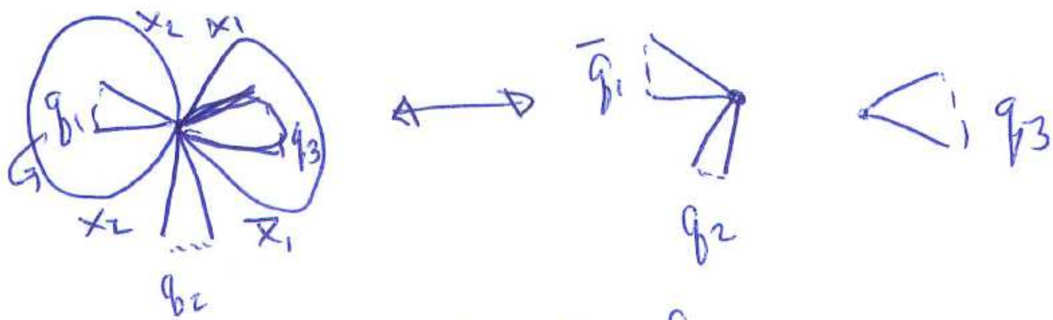
This corresponds to the reduction of the Möbius graph given in Figure 18 and represents a net of 3 to the Euler characteristic.

(3) When $X_1 = \bar{X}_k$ and $X_2 = X_j$ we find that

$$\begin{aligned}
&\mathbb{E}(X_1 X_2 q_1 X_2 q_2 \bar{X}_1 q_3)\\
&= \mathbb{E}(X_1 \bar{X}_1)\mathbb{E}(\Re(X_2 q_1 X_2 q_2)\Re(q_3))\\
&= 4\mathbb{E}(X_2 X_2)\mathbb{E}(\Re(\bar{q}_1 q_2)\Re(q_3))\\
&= (-2)^3\left[\mathbb{E}(\Re(\bar{q}_1 q_2)\Re(q_3))\right]\\
&= (-2)^3\left[(-2)^{\chi_2} 4^{(n-2)-2}\right]\\
&= (-2)^{\chi_2 - 3} 4^{n-1}\,.
\end{aligned}$$

This corresponds to the reduction for the Möbius graph given in Figure 19, represents a net of 3 to the Euler characteristic, and the result follows.

FIGURE 19.

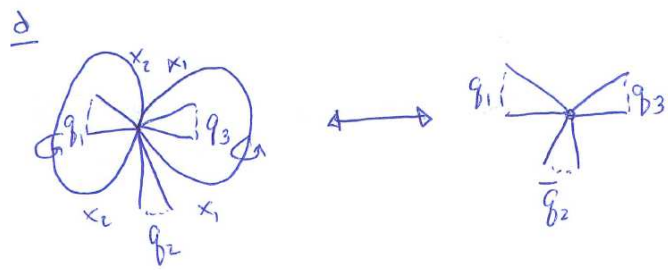


$$\chi_2 = (v_1 + 1) - (e_1 - 2) + f_1$$
$$= \chi_1 + 3$$

(4) When $X_1 = X_k$ and $X_2 = X_j$ we find that

$$
\begin{aligned}
&\mathbb{E}(X_1 X_2 q_1 X_2 q_2 X_1 q_3)\\
&= \mathbb{E}(X_1 X_1)\mathbb{E}(\bar{q}_2 \bar{X}_2 \bar{q}_1 \bar{X}_2 q_3)\\
&= (-2)\mathbb{E}(\bar{X}_2 \bar{q}_1 \bar{X}_2 q_3 \bar{q}_2)\\
&= (-2)\mathbb{E}(\bar{X}_2 \bar{X}_2)\mathbb{E}(q_1 q_3 \bar{q}_2)\\
&= (-2)^2 \mathbb{E}(q_1 q_3 \bar{q}_2)\\
&= (-2)^2 \left[(-2)^{\chi_2} 4^{(n-2)-1}\right]\\
&= (-2)^{\chi_2 - 2} 4^{n-1}.
\end{aligned}
$$

FIGURE 20.



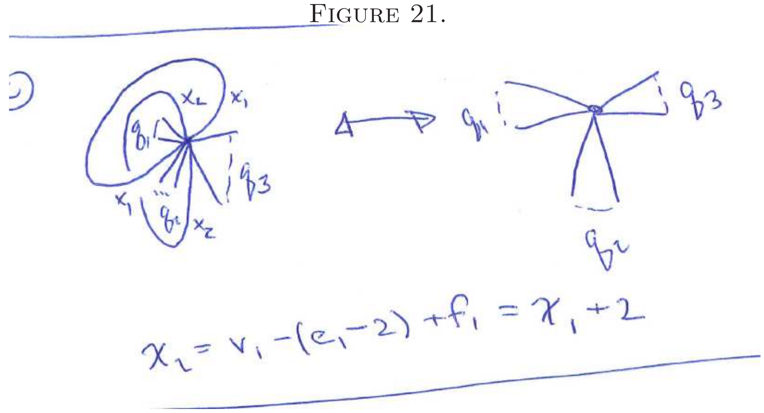
$$\chi_2 = v_1 - (e_1 - 2) + f_1 = \chi_1 + 2$$

This corresponds to the reduction for the Möbius graph given in Figure 20, represents a net of 2 to the Euler characteristic ($\chi_2 = \chi_1 + 2$), and the result follows.

(5) When $X_1 = \bar{X}_j$ and $X_2 = \bar{X}_k$ we find that

$$\begin{aligned}
&\Re\mathbb{E}(X_1 X_2 q_1 \bar{X}_1 q_2 \bar{X}_2 q_3)\\
&= \mathbb{E}(X_1\bar{X}_1)\mathbb{E}(\Re(X_2 q_1)\Re(q_2 \bar{X}_2 q_3))\\
&= 4\mathbb{E}(\Re(X_2 q_1)\Re(\bar{X}_2 q_3 q_2))\\
&= 4\mathbb{E}(\Re(q_1 q_3 q_2))\\
&= 4\left[(-2)^{\chi_2} 4^{(n-2)-1}\right]\\
&= (-2)^{\chi_2 - 2} 4^{n-1}\,.
\end{aligned}$$

FIGURE 21.



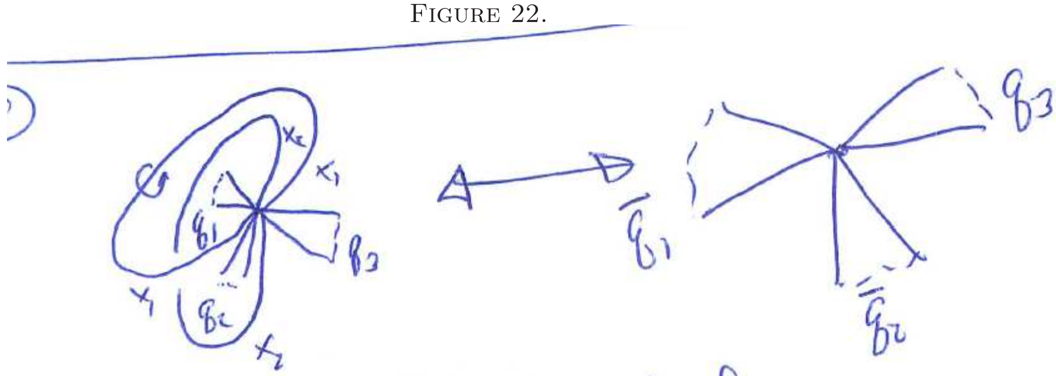
This corresponds to the reduction for the Möbius graph given in Figure 21, represents a net of 2 to the Euler characteristic, and the result follows.

(6) When $X_1 = X_j$ and $X_2 = \bar{X}_k$ we find that

$$\begin{aligned}
&\mathbb{E}(X_1 X_2 q_1 X_1 q_2 \bar{X}_2 q_3)\\
&= \mathbb{E}(X_1 X_1)\mathbb{E}(\bar{q}_1 \bar{X}_2 q_2 \bar{X}_2 q_3)\\
&= (-2)\mathbb{E}(\bar{X}_2 q_2 \bar{X}_2 q_3 \bar{q}_1)\\
&= (-2)\mathbb{E}(\bar{X}_2 \bar{X}_2)\mathbb{E}(\bar{q}_2 q_3 \bar{q}_1)\\
&= (-2)^2\mathbb{E}(\bar{q}_2 q_3 \bar{q}_1)\\
&= (-2)^{-2}\left[(-2)^{\chi_2} 4^{(n-2)-1}\right]\\
&= (-2)^{\chi_2 - 2} 4^{n-1}\,.
\end{aligned}$$

This corresponds to the reduction for the Möbius graph given in Figure 22, represents a net of 2 to the Euler characteristic, and the result follows.

FIGURE 22.



$$\chi_2 = v_1 - (e_1 - 2) + f_1$$
$$= \chi_1 + 2$$

(7) When $X_1 = \bar{X}_j$ and $X_2 = X_k$ we find that

$$\begin{aligned}
&\mathbb{E}(X_1 X_2 q_1 \bar{X}_1 q_2 X_2 q_3) \\
&= \mathbb{E}(X_1 \bar{X}_1)\mathbb{E}(\Re(X_2 q_1)\Re(q_2 X_2 q_3)) \\
&= 4\mathbb{E}(\Re(X_2 q_1)\Re(X_2 q_3 q_2)) \\
&= 4\mathbb{E}(\bar{q}_1 q_3 q_2) \\
&= 4\left[(-2)^{\chi_2} 4^{(n-2)-1}\right] \\
&= (-2)^{\chi_2 - 2} 4^{n-1}\,.
\end{aligned}$$

This corresponds to the reduction for the Möbius graph given in Figure 23, represents a net of 2 to the Euler characteristic, and the result follows.

(8) When $X_1 = X_j$ and $X_2 = X_k$ we find that

$$\begin{aligned}
&\mathbb{E}(X_1 X_2 q_1 X_1 q_2 X_2 q_3) \\
&= \mathbb{E}(X_1 X_1)\mathbb{E}(\bar{q}_1 \bar{X}_2 q_2 X_2 q_3) \\
&= (-2)\mathbb{E}(X_2 q_3 \bar{q}_1 \bar{X}_2 q_2) \\
&= (-2)\mathbb{E}(X_2 \bar{X}_2)\mathbb{E}(\Re(q_3 \bar{q}_1)\Re(q_2)) \\
&= (-2)^3\mathbb{E}(\Re(q_3 \bar{q}_1)\Re(q_2)) \\
&= (-2)^3\left[(-2)^{\chi_2} 4^{(n-2)-2}\right] \\
&= (-2)^{\chi_2 - 3} 4^{n-1}\,.
\end{aligned}$$

This corresponds to the reduction for the Möbius graph given in Figure 24, represents a net of 3 to the Euler characteristic, and the result follows.

FIGURE 23.

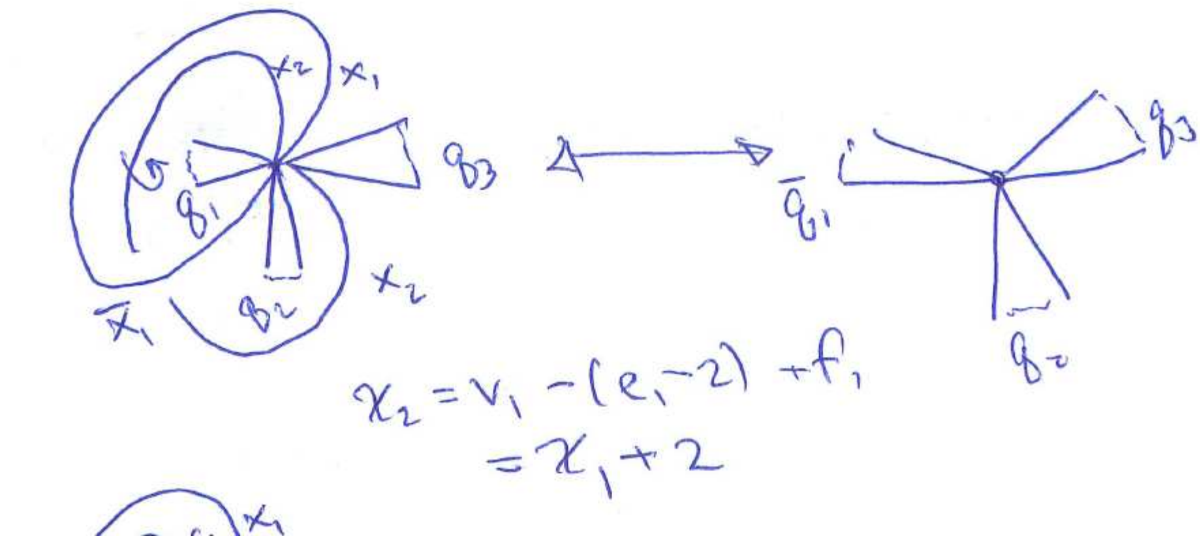
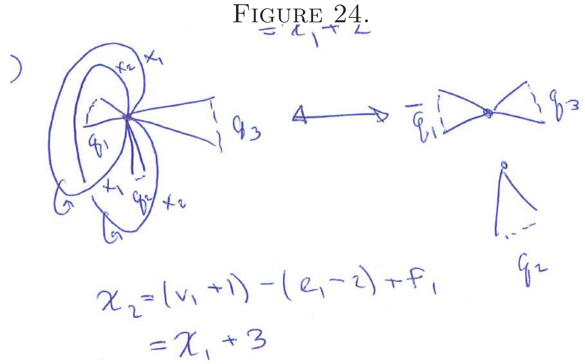


FIGURE 24.



Note that taking away an oriented ribbon creates a new vertex. The remaining graph might be still connected, or it may split into two components. If taking away a loop makes the graph disconnected, then the counts of changes to edges and vertices are still the same. But the faces need to be counted as follows: The inner face of the removed edge becomes the outside face of one component, and the outer face at the removed edge becomes the outer face of the other component. Thus the counting of faces is not affected by whether the graph is connected – no change.

With each of these cases checked, by the induction hypothesis, the proof is completed.

□

4. Duality between GOE and GSE ensembles

The duality between $\beta = 1$ and $\beta = 4$ asserts that the formulas for moments of traces of polynomials in $\mathbb{H}$-valued Gaussian ensembles can be obtained from the corresponding formulas for the $\mathbb{R}$-valued Gaussian ensembles by replacing the dimension parameter N by $(-2N)$. The duality for one-matrix Wishart ensembles was discovered in [1, Corollary 4.2]. The duality for one-matrix GOE/GSE ensembles appears in [5]. A derivation based on recurrence formulas for moments is in [3].

4.1. **Quaternion matrices.**

By $\mathcal{M}_{M\times N}(\mathbb{H}_\beta)$ we denote the set of all $M \times N$ matrices with entries from $\mathbb{H}_\beta$. For $\mathbf{A} \in \mathcal{M}_{M\times N}(\mathbb{H}_\beta)$, the conjugate matrix is $A^*_{i,j} := \overline{A}_{j,i}$. The trace is $\mathrm{tr}(\mathbf{A}) = \sum_{j=1}^N A_{jj}$.

When $\beta = 4$, traces $\mathrm{tr}(\mathbf{A})$ may fail to commute, so in the formulas we will use $\Re(\mathrm{tr}(\mathbf{A}))$. In fact, the natural definition of trace would be $\mathrm{tr}_\beta(\cdot) := \beta\Re(\mathrm{tr}(\cdot))$. Such a nonstandard definition is consistent with the treatment of $\mathbf{A} \in \mathcal{M}_{N\times N}(\mathbb{H}_\beta)$ as an element of $\mathcal{M}_{\beta N\times\beta N}(\mathbb{R})$ via transformation (**??**). Non-standard definition of quaternionic trace is used in [1]

4.1.1. *GSE Ensembles.*

We will be interested GUE_β ensembles. These are square self-adjoint matrices

$$\mathbf{Z} = [Z_{i,j}] \tag{4.1}$$

where $\{Z_{i,j} : i < j\}$ is a family of independent $\mathbb{H}_\beta$-normal random variables, $\{Z_{i,i}\}$ is a family of independent real normal of variance 2, and $Z_{i,j} = \bar{Z}_{j,i}$. The law of such a matrix has a density $C\exp(-\frac{1}{2}\mathrm{tr}(\mathbf{x}^2))$, which is supported on the self-adjoint subset of $\mathcal{M}_{N\times N}(\mathbb{H}_\beta)$. ($C = C(N, \beta)$ is a normalizing constant.)

4.2. **A multi-matrix version of Mulase and Waldron.**

In this section we will demonstrate a multi-matrix version of the theorem of Mulase and Waldron [5] from a heuristic argument using our Wick formula and quadratic relations. This represents an improvement to the argument of Mulase and Waldron as we will not need to rely on labelings of the vertices by the quaternions $1, i, j$, and k. This part of their argument is encoded in relations (2.5) and (2.6).

We will compute here the expected values of the real traces of powers of a quaternionic self-dual matrix in the Gaussian symplectic ensemble (4.1) where the off-diagonal matrix entries are (2.2).

The basic theorem we will prove is

Theorem 4.1.

$$\frac{1}{(4N)^{n-m}}\mathbb{E}(\Re(\mathrm{tr}(\mathbf{Z}^{j_1}))\Re(\mathrm{tr}(\mathbf{Z}^{j_2}))\dots\Re(\mathrm{tr}(\mathbf{Z}^{j_m}))) = \sum_\Gamma (-2N)^{\chi(\Gamma)},$$

where the sum is over labeled Möbius graphs Γ with m vertices of degree $j_1, j_2, \dots, j_m$, $\chi(\Gamma)$ is the Euler characteristic and $j_1 + j_2 + \dots + j_m = 2n$.

More generally, suppose $\mathbf{Z}_1, \dots, \mathbf{Z}_s$ are independent $N \times N$ GSE ensembles and $t : \{1, \dots, 2n\} \to \{1, \dots, s\}$ is fixed. Let $\alpha_0 = 1$, $\alpha_k = j_1 + j_2 + \dots + j_{k-1} + 1$,

$\beta_k = j_1 + j_2 + \cdots + j_k$ denote the ranges under traces. Then

(4.2)
$$\frac{1}{(4N)^{n-m}}\mathbb{E}\left(\Re(\mathrm{tr}(\mathbf{Z}_{t(1)}\dots\mathbf{Z}_{t(\beta_1)}))\Re(\mathrm{tr}(\mathbf{Z}_{t(\alpha_2)}\dots\mathbf{Z}_{t(\beta_2)}))\dots\Re(\mathrm{tr}(\mathbf{Z}_{t(\alpha_m)}\dots\mathbf{Z}_{t(\beta_m)}))\right)$$
$$= \sum_\Gamma (-2N)^{\chi(\Gamma)},$$

where the sum is over labeled color-preserving Möbius graphs Γ with vertices of degree $j_1, j_2, \dots, j_m$ that are colored with s colors by the mapping t. As previously, $\chi(\Gamma)$ is the Euler characteristic and $j_1 + j_2 + \cdots + j_m = 2n$. (If there are no Γ that are consistent with the coloring t, we interpret the sum as 0.)

Perhaps instead of the example, we should clarify how come the Theorem IS the "duality"?

Example 4.2. To illustrate the multi-matrix aspect of the theorem, suppose $\mathbf{Z}_1$ and $\mathbf{Z}_2$ are independent $N\times$ GSE and $\mathbf{X}_1, \mathbf{X}_2$ be independent $N\times$ GOE ensembles. Fix r, m, k and let $f(N) := \mathbb{E}(\mathrm{tr}(\mathbf{X}_1^{2m}\mathbf{X}_2^{2k})^r)$. Since $f(N) = N^{(m+k-1)r}\sum_\Gamma N^{\chi(\Gamma)}$, [reference?] therefore Theorem 4.1 implies that the moments for the independent GSE ensembles are determined from the corresponding moments of independent GOE ensembles by the "dual formula"
$$\mathbb{E}(\mathrm{tr}(\mathbf{Z}_1^{2m}\mathbf{Z}_2^{2k})^r) = (4)^{(m+k-1)r} f(-2N).$$

Proof. We begin by expanding out the traces in terms of the matrix entries

(4.3)
$$\frac{N^m}{4^{n-m}}\mathbb{E}(\Re(\mathrm{tr}(\mathbf{Z}^{j_1}))\Re(\mathrm{tr}(\mathbf{Z}^{j_2}))\dots\Re(\mathrm{tr}(\mathbf{Z}^{j_m}))) =$$
$$= \sum_{\substack{1\le a_1, a_2, \dots, a_{j_1}\le N\\ 1\le b_1, b_2, \dots, b_{j_2}\le N\\ \vdots\\ 1\le c_1, c_2, \dots, c_{j_m}\le N}} \begin{matrix}\frac{1}{(4n)^{n-m}}\mathbb{E}(\Re(Z_{a_1,a_2}Z_{a_2,a_3}\dots Z_{a_{j_1},a_1})\Re(Z_{b_1,b_2}\dots Z_{b_{j_2},b_1})\dots\\ \dots\Re(Z_{c_1,c_2}\dots Z_{c_{j_m},c_1}))\,.\end{matrix}$$

Note that essentially the same expansion applies to (4.2), except that the consecutive entries $Z_{i,j}^{(t)}$ must now be labeled also by the "color" t.

Colors make for cumbersome notation! It is then better to index the products by the cycles of a permutation, so that the formula is
$$\sum_{a:[1\dots 2n]\to[1\dots N]}\prod_{c\in\sigma}\mathbb{E}(\Re(\prod_{j\in c} Z^{t(j)}_{a(j),a(\sigma(j))})).$$
We then need a special convention to follow the right order for products over the cycles. Luckily, $\Re$ of quaternions has cyclic property so such products are well defnied.

From (2.8) it follows that the right hand side of (4.3) can be expanded as sum over all pairings, and we can assume that the pairs are independent.

Of course, pairings that match two independent random variables do not contribute to the sum. The pairings that contribute to the sum are of three different types: pairs that match Z with another Z at a different position in the product, pairs that match Z with $\bar{Z}$, and pairings that match the diagonal (real) entries.

We first dispense with the pairings that match a diagonal entry $Z_{i,i} = \xi$ with another diagonal entry $Z_{j,j} = \xi$. Since real numbers commute with quaternions,

$$\Re(\mathbb{E}(q_0\xi q_1\xi q_2)) = \mathbb{E}(\xi^2)\Re(q_0q_1q_2) = 2\Re(q_0q_1q_2).$$

On the other hand, using the cyclic property of $\Re(\cdot)$ and adding formulas (2.3) and (2.4), we see that the same answer arises from

$$\mathbb{E}\Re(q_0Zq_1Zq_2) + \mathbb{E}\Re(q_0Zq_1\bar{Z}q_2) = \mathbb{E}\Re(Zq_1Zq_2q_0) + \mathbb{E}\Re(Zq_1\bar{Z}q_2q_0) = 2\Re(q_1q_2q_0).$$

So the contribution of each such diagonal pairing is the same as that of two matches of quaternionic entries (Z, Z) and $(Z, \bar{Z})$.

Thus, once we replace all the real entries that came from the diagonal entries by the corresponding quaternion-normal pairs, we get the sum over all possible pairs of matches of the first two types only. We label all such pairings by Möbius graphs, with the interpretation that pairings or random variables Z, Z correspond to twisted ribbons, while pairings Z, $\bar{Z}$ correspond to ribbons without a twist. The pairs of variables at different ribbons can now be assumed independent. In the multi-matrix case, the edges of the graph at each vertex are colored according to function t, which restricts the number of available pairings.

We now relabel the $Z_{i,j}$ by $Z_{a_k,a_{k+1}} = X_k$, $Z_{b_k,b_{k+1}} = X_{j_1+k}$, $\dots$, $M_{c_k,c_{k+1}} = X_{j_1+j_2+\dots j_{m-1}+k}$. Our claim is then that given a Möbius graph Γ with vertices of degree j_1, $j_2, \dots$, and j_m with edges labeled by X_k, satisfies

$$\frac{1}{(4N)^{n-m}}\mathbb{E}(\Re(X_1\dots X_{j_1})\Re(X_{j_1+1}\dots X_{j_1+j_2})\dots\Re(X_{j_1+\dots+j_{m-1}+1}\dots X_{2n})) \\ = (-2N)^{\chi(\Gamma)}N^{-f(\Gamma)}\,, \tag{4.4}$$

where $f(\Gamma)$ is the number of faces of Γ. The $N^{-f(\Gamma)}$ terms are removed by the summations in (4.3), the relations given by the edges of Γ reduce the number of summations leaving us with $f(\Gamma)$ sums from 1 to N. Collecting the powers of N we find that this is the same as (3.2). $\square$

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