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MOMENT CONDITIONS FOR ALMOST SURE CONVERGENCE OF WEAKLY CORRELATED RANDOM VARIABLES

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ABSTRACT. For random sequences with unrestricted maximal correlation coefficient strictly less than 1, sufficient moment conditions for almost sure convergence of a series and for the strong law of large numbers are given.

NOTATION AND RESULTS

Suppose $(\xi_k)_{k\in\mathbb{N}}$ is a real random sequence on a probability space $(\Omega, \mathscr{M}, P)$. For $S \subset \mathbb{N}$ define σ-fields $\mathscr{F}_S = \sigma\{\xi_k : k \in S\}$. Given σ-fields $\mathscr{F}$, $\mathscr{G}$ in $\mathscr{M}$ let

$$\rho(\mathscr{F}, \mathscr{G}) := \sup\{\operatorname{corr}(V; W) : V \in L_2(\mathscr{F}),\ W \in L_2(\mathscr{G})\}.$$

Following Bradley [3] for $k \geq 0$ we define the following coefficients of dependence:

$$\tilde{\rho}(k) := \sup\{\rho(\mathscr{F}_S, \mathscr{F}_T)\}, \tag{1}$$

where the supremum is taken over all finite subsets $S, T \subset \mathbb{N}$ such that $\operatorname{dist}(S, T) \geq k$. Clearly, $0 \leq \tilde{\rho}(k+1) \leq \tilde{\rho}(k) \leq 1$, $k \geq 0$, and $\tilde{\rho}(0) = 1$.

Definition (1) resembles the definition of the so-called *maximal correlation coefficient,* which is defined by (1) with index sets restricted to subsets S of $[1, n]$ and subsets T of $[n+k, \infty)$, n, $k \in \mathbb{N}$. The maximal correlation coefficient is usually denoted in the literature by $\rho(k)$; Bradley [2, 3] denotes by $\rho^*(r)$ what in our notation is $\tilde{\rho}(r)$. Conditions such as $\tilde{\rho}(k) \to 0$ (as $k \to \infty$) have been used in the study of weak limit theorems for random fields, see Bradley [2, 3] and the references therein.

We would like to point out explicitly that the condition $\tilde{\rho}(k) \to 0$ follows from the following *hypercontractivity* condition.

(H) There exist $q(k) \to \infty$ (as $k \to \infty$) such that if $S, T \subset \mathbb{N}$ satisfy $\operatorname{dist}(S, T) \geq k$ then the norm of conditional expectation $E\{\ |\mathscr{F}_S\}$ as a linear operator from $L_2(\mathscr{F}_T)$ to $L_{q(k)}(\mathscr{F}_S)$ is 1.

This can be easily seen from Deuschel and Stroock [6, Lemma 5.5.11]; their proof gives $\tilde{\rho}(k) << 1/\sqrt{q(k)-1}$ as $k \to \infty$.

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The purpose of this note is to show that the condition $\lim_{k\to\infty} \tilde{\rho}(k) < 1$ suffices to get the criterion for almost sure convergence of a random series in terms of the moments of individual random variables. We also prove the strong law of large numbers under a somehow weaker mixing condition. For other almost sure convergence results using different mixing measures of dependence, see Peligrad [9], Stout [11], and the references therein.

Theorem 1. *Suppose* $\tilde{\rho}(k) < 1$ *for some* $k < \infty$. *Assume* $\mathrm{E}\{\xi_k\} = 0$, $\mathrm{E}\{\xi_k^2\} = 1$ *for all* k, *and suppose there is* $\delta > 0$ *such that* $\sup_k \mathrm{E}\{|\xi_k|^{2+\delta}\} < \infty$. *If* $\sum_{k=1}^{\infty} a_k^2 < \infty$, *then the series* $\sum_{k=1}^{\infty} a_k \xi_k$ *converges almost surely.*

By Kronecker's lemma, Theorem 1 implies the law of large numbers. However, using Szablowski [12], a stronger result can be obtained, assuming both weaker mixing and weaker moment conditions. The weaker mixing condition is based on the following coefficient. For $k \geq 0$, let

$$\tilde{r}(k) = \sup\{\operatorname{corr}(V; W)\},$$

where the supremum is taken over all finite subsets $S, T \subset \mathbb{N}$ such that $\operatorname{dist}(S, T) \geq k$ and over all linear combinations V of variables $\{\xi_k\}_{k\in S}$ and all linear combinations W of variables $\{\xi_k\}_{k\in T}$.

Clearly, $\tilde{r}(k) \leq \tilde{\rho}(k)$; $\tilde{r}(1) = 0$ iff $\{\xi_k\}$ are uncorrelated; and $\tilde{r}(k)$ is a nonincreasing function of k. For an L_2-stationary sequence, condition $\tilde{r}(k) \to 0$ as $k \to \infty$ implies that $\{\xi_k\}$ has a continuous spectral density; $[\tilde{r}(1) < 1$ and $\tilde{r}(k) \to 0]$ is equivalent to $\{\xi_k\}$ having continuous and positive spectral density, see Bradley [2, Theorem 2]. Spectral density conditions equivalent to either the exponential or polynomial rate of convergence of $\tilde{r}(k)$ to 0 are given in Cheng [5, Theorem 1.1].

The following result resembles the well-known martingale law of large numbers (cf. Feller [7, Chapter VII.8]), improves upon a direct application of Kronecker's lemma to Theorem 1 and extends Szablowski [12, Corollary 9] to the "weakly orthogonal case".

Theorem 2. *If* $\tilde{r}(k) < 1$ *for some* k, $\mathrm{E}\{\xi_k\} = 0$, *for all* k, *and*

$$\sum_k k^{-3/2}\mathrm{E}\{\xi_k^2\} < \infty, \tag{2}$$

then $\frac{1}{n}\sum_{k=1}^{n} \xi_k \to 0$ *almost surely.*

Remark 1. For simplicity of notation, throughout this note we consider $\mathbb{N}$-indexed random sequences only. Theorem 1 holds true for the d-dimensional index set $\mathbb{Z}^d$, $d \geq 1$, too; the only change needed in the proof is to use Moricz [8, Theorem 7] instead of Lemma A below.

Remark 2. Under the assumptions of Theorem 1, one can show that $\sum_{k=1}^{\infty} a_k \xi_k$ converges in $L_{2+\delta}$. If $0 \leq \delta < 1$, this is established in the course of the proof below; some modifications are needed to cover the case when $\delta \geq 1$.

Remark 3. Even in the stationary case, it may happen that $\tilde{\rho}(1) < 1$ while $\lim_{k\to\infty} \tilde{\rho}(k) \neq 0$; take a gaussian sequence with the noncontinuous, positive, bounded, and bounded away from zero spectral density, see the proof of Theorem 2 in Bradley [2].

PROOFS

Throughout the proofs we use the same symbol C for various constants that may depend on the value of $\tilde{\rho}(1)$; $\|\cdot\|_p$ denotes the L_p-norm.

Let $a_1, a_2, \ldots$ be fixed. In what follows $X_k = a_k\xi_k$ and Y_k denotes arbitrary centered $\sigma(\xi_k)$-measurable random variable, $k \geq 1$. Theorem 1 asserts that $\sum_{k=1}^{\infty} X_k$ converges almost surely.

Without loss of generality we may assume that (X_k) is such that $\tilde{\rho}(1) < 1$. Indeed, if $\tilde{\rho}(N) < 1$ for some $N > 1$, then $\sum_{k=1}^{\infty} X_k$ can be written as the sum of N terms of the form $\sum_{k=0}^{\infty} X_{Nk+j}$, $j = 1, 2, \ldots, N$, and the coefficient $\tilde{\rho}(1)$ defined for $(X_{Nk+j})_{k\in\mathbb{N}}$ is dominated by $\tilde{\rho}(N)$.

Lemma A. *If for some $\delta > 0$, $C < \infty$, all $N \geq 1$, and all $k \leq N$,*

$$\mathrm{E}\left\{\left|\sum_{j=k}^{N} X_j\right|^{2+\delta}\right\} \leq C\left(\sum_{j=k}^{N} a_j^2\right)^{1+\delta/2}, \tag{3}$$

then there is $K < \infty$ such that for all $N \geq 1$, $t > 0$,

$$P\left(\max_{k\leq N}\left|\sum_{j=1}^{k} X_j\right| \geq t\right) \leq \frac{K}{t^{2+\delta}}\left(\sum_{j=1}^{N} a_j^2\right)^{1+\delta/2}.$$

This lemma is known, see Billingsley [1, Chapter 2, Theorem 12.2].

The following result is essentially contained in [4].

Lemma 1. *Fix $1 < p < \infty$ and σ-fields $\mathcal{F}$, $\mathcal{F}_*$ in $\mathcal{M}$. If $\rho = \rho(\mathcal{F}, \mathcal{F}_*) < 1$ then there is $C = C(p, \rho)$ such that the following implication holds. If X (respectively, X_*) is a centered $\mathcal{F}$-measurable (respectively, $\mathcal{F}_*$-measurable) random variable, $\mathrm{E}\{|X|^p\} < \infty$ and $\mathrm{E}\{|X_*|^p\} < \infty$, then there is a random variable Z on the same probability space $(\Omega, \mathcal{M}, P)$ such that*

$$X = \mathrm{E}\{Z|\mathcal{F}\} \quad \text{and} \quad X_* = \mathrm{E}\{Z|\mathcal{F}_*\}; \tag{4}$$

$$\|Z\|_p \leq C(\|X\|_p + \|X_*\|_p). \tag{5}$$

Remark 4. For the proof of Theorem 1, we shall need only values $1 < p \leq 2$. In this case our proof gives $C(p, \rho) = 2^{2/p}(1 - \rho^{2-2/p})^{-1}$.

Proof. Suppose that $1 < p \leq 2$. Let A be a linear operator defined by $A(\cdot) = \mathrm{E}\{\mathrm{E}\{\cdot|\mathcal{F}_*\}|\mathcal{F}\}$ and let $E(\cdot)$ denote the linear operator of taking the expected value $\mathrm{E}\{\cdot\}$. Clearly, $\|A - \mathrm{E}\|_{L_2\to L_2} \leq \rho$ and $\|A^k - \mathrm{E}\|_{L_1\to L_1} \leq 2$. Furthermore, it is easy to check that for $k \geq 1$ we have $(A - \mathrm{E})^k = A^k - \mathrm{E}$. Therefore, for $k \geq 1$, the Riesz interpolation theorem (here we use the real version of the Riesz-Thörin theorem, see [10, §22]) implies

$$\|A^k - \mathrm{E}\|_{L_p\to L_p} \leq (\|(A-\mathrm{E})^k\|_{L_2\to L_2})^{2-2/p}(\|A^k - \mathrm{E}\|_{L_1\to L_1})^{2/p-1} \leq \rho^{2k(1-1/p)}2^{2/p-1}.$$

In particular, for centered Y we have $\|A^kY\|_p \leq 2^{2/p-1}\rho^{2k(1-1/p)}\|Y\|_p$. Since the estimate also holds true for $k = 0$ and, by symmetry, also for $A^*(Y) = \mathrm{E}\{\mathrm{E}\{Y|\mathcal{F}\}|\mathcal{F}_*\}$,

$$Z = \sum_{k=0}^{\infty} A^k(X - \mathrm{E}\{X_*|\mathcal{F}\}) + \sum_{k=0}^{\infty}(A^*)^k(X_* - \mathrm{E}\{X|\mathcal{F}_*\})$$

is well defined and (5) holds by the triangle inequality. By a simple computation (4) holds. This ends the proof in the case $1 < p \le 2$. If $2 < p < \infty$ the only change needed in the proof is to use $\|A^k - \mathrm{E}\|_{L_\infty \to L_\infty} \le 2$ for the Riesz-Thörin interpolation theorem.

The following result is related to Lemma 1 and Lemma 5 in Bradley [2].

Lemma 2. *Let $\tilde{\rho} := \tilde{\rho}(1) < 1$, $q \ge 1$, be fixed. There is $C = C(q, \tilde{\rho}) < \infty$, which depends on $\tilde{\rho}$ and q only such that if Y_j are centered $\sigma(\xi_j)$-measurable with finite qth moments, $j \ge 1$, then for all $N \ge 1$ and all $k \le N$,*

$$\mathrm{E}\left\{\left|\sum_{j=k}^{N} Y_j\right|^q\right\} \le C\mathrm{E}\left\{\left(\sum_{j=k}^{N} Y_j^2\right)^{q/2}\right\}. \tag{6}$$

Proof. Fix $k \le N$. Let (ε_j) be i.i.d. random variables independent of (Y_j) with $P(\varepsilon_j = \pm 1) = 1/2$. We claim that there is $K = K(q, \tilde{\rho}) < \infty$ such that

$$\mathrm{E}\left\{\left|\sum_{j=k}^{N} Y_j\right|^q\right\} \le K\mathrm{E}\left\{\left|\sum_{j=k}^{N} \varepsilon_j Y_j\right|^q\right\}. \tag{7}$$

Clearly, (6) follows from (7) by the Khinchine inequality.

To prove (7), following Bradley [2] we consider random variables S, S_* on the product probability space $2^{\{k, k+1, \dots, N\}} \times \Omega$ with the product measure $P_1(Q \times A) = P(A)/2^{N-k+1}$, $Q \subset \{k, k+1, \dots, N\}$; S and S_* are defined by

$$S(Q, \omega) = S_Q(\omega) = \sum_{j \in Q} Y_j(\omega), \qquad S_*(Q, \omega) = S_{Q^*}(\omega) = \sum_{j \in Q^*} Y_j(\omega),$$

where $Q^* = \{k, k+1, \dots, N\} \backslash Q$.

Clearly, for each Q and every $\omega \in \Omega$ we have $S_Q + S_{Q^*} = \sum_{j=k}^{N} Y_j$. In particular,

$$\left\|\sum_{j=k}^{N} Y_j\right\|_q \le \|S_Q\|_q + \|S_{Q^*}\|_q. \tag{8}$$

Moreover, by a well-known consequence of the Hahn-Banach theorem applied to each of the terms on the right-hand side of (8) separately, we get

$$\|S_Q\|_q + \|S_{Q^*}\|_q = \mathrm{E}\{Y S_Q\} + \mathrm{E}\{Y_* S_{Q^*}\}, \tag{9}$$

where Y (respectively, Y_*) is $\mathscr{F}_Q$-measurable (respectively, $\mathscr{F}_{Q^*}$-measurable) and

$$\mathrm{E}\{|Y_*|^p\} = \mathrm{E}\{|Y|^p\} = 1, \qquad 1/p + 1/q = 1.$$

Since by assumption $\mathrm{E}\{S_Q\} = 0$, we can replace Y and Y_* in (9) by centered variables

$$X = Y - \mathrm{E}\{Y\}, \qquad X_* = Y_* - \mathrm{E}\{Y_*\}.$$

For each fixed Q, by Lemma 1 applied to σ-fields $\mathscr{F} = \mathscr{F}_Q$, $\mathscr{F}_* = \mathscr{F}_{Q^*}$, and random variables X, $-X_*$, there is Z such that

$$\|Z\|_p \le 4C(p, \tilde{\rho}), \qquad \mathrm{E}\{Z|\mathscr{F}_Q\} = X, \qquad \mathrm{E}\{Z|\mathscr{F}_{Q^*}\} = -X_*.$$

Therefore

$$\begin{aligned}\mathrm{E}\{YS_Q\} + \mathrm{E}\{Y_* S_{Q^*}\} &= \mathrm{E}\{XS_Q\} - E\{(-X_*)S_{Q^*}\} = \mathrm{E}\{ZS_Q\} - \mathrm{E}\{ZS_{Q^*}\} \\ &\le \|Z\|_p \|S_Q - S_{Q^*}\|_q .\end{aligned}$$

This shows that there is $K = (4C(p, \tilde{\rho}))^q$ such that for all $Q \subset \{k, k+1, \dots, N\}$ we have

$$\mathrm{E}\left\{\left|\sum_{j=k}^{N} Y_j\right|^q\right\} \le K\mathrm{E}\{|S_Q - S_{Q^*}|^q\}.$$

Since in distribution $S - S_* \simeq \sum_{j=k}^{N} \varepsilon_j Y_j$, averaging the last inequality over all subsets $Q \subset \{k, k+1, \dots, N\}$ gives (7).

Lemma 3. *Let $2 \le q \le 4$ be fixed. If $\tilde{\rho} := \tilde{\rho}(1) < 1$ then there is $C = C(q, \tilde{\rho}) < \infty$, which depends on $\tilde{\rho}$ and q only and such that for all $N \ge 1$ and all $k \le N$*

$$\mathrm{E}\left\{\left|\sum_{j=k}^{N} X_j\right|^q\right\} \le C\left(\left(\sum_{j=k}^{N} \mathrm{E}\{X_j^2\}\right)^{q/2} + \sum_{j=k}^{N} \mathrm{E}\{|X_j|^q\}\right). \tag{10}$$

Proof. Fix $k \le N$. Let $U_j = X_j^2 - \mathrm{E}\{X_j^2\}$. By the triangle inequality

$$\left|\sum_{j=k}^{N} X_j^2\right| \le \left|\sum_{j=k}^{N} U_j\right| + \sum_{j=k}^{N} \mathrm{E}\{X_j^2\},$$

hence (6) applied to $Y_j = X_j$ gives

$$\mathrm{E}\left\{\left|\sum_{j=k}^{N} X_j\right|^q\right\} \le C'\left(\mathrm{E}\left\{\left|\sum_{j=k}^{N} U_j\right|^{q/2}\right\} + \left(\sum_{j=k}^{N} \mathrm{E}\{X_j^2\}\right)^{q/2}\right). \tag{11}$$

By (6) applied to $Y_j = U_j$, we have

$$\mathrm{E}\left\{\left|\sum_{j=k}^{N} U_j\right|^{q/2}\right\} \le C''\mathrm{E}\left\{\left(\sum_{j=k}^{N} U_j^2\right)^{q/4}\right\}.$$

Since $q \le 4$, this and (11) imply

$$\mathrm{E}\left\{\left|\sum_{j=k}^{N} X_j\right|^q\right\} \le C\mathrm{E}\left\{\sum_{j=k}^{N} |U_j|^{q/2}\right\} + C\left(\sum_{j=k}^{N} \mathrm{E}\{X_j^2\}\right)^{q/2}.$$

The trivial inequality $\|U_j\|_{q/2} \le 2(\|X_j\|_q)^2$ now ends the proof of (10).

Proof of Theorem 1. Without loss of generality we may assume $0 < \delta \le 2$. By Lemma 3 with $q = 2 + \delta$, there is C such that for all $N \ge 1$ and all $k \le N$

$$\mathrm{E}\left\{\left|\sum_{j=k}^{N} X_j\right|^{2+\delta}\right\} \le C\left(\sum_{j=k}^{N} \mathrm{E}\{X_j^2\}\right)^{1+\delta/2} + C\sum_{j=k}^{N} \mathrm{E}\{|X_j|^{2+\delta}\}. \tag{12}$$

From (12) we easily get (3). Indeed, $\mathrm{E}\{X_j^2\} = a_j^2$ and

$$\sum_{j=k}^{N} \mathrm{E}\{|X_j|^{2+\delta}\} \le \sup_j \mathrm{E}\{|\xi_j|^{2+\delta}\} \sum_{j=k}^{N} |a_j|^{2+\delta} \le C \left(\sum_{j=k}^{N} a_j^2 \right)^{1+\delta/2} .$$

By the definition of almost sure convergence, Theorem 1 now follows from Lemma A.

Proof of Theorem 2. The result is an immediate consequence of Theorem 4 in Szablowski [12] used together with Lemma 1 in Bradley [2]. Since in our case there is no need for using the full scope of Szablowski's theory, a short proof based on his ideas is given below.

Proof. As in the proof of Theorem 1, without loss of generality we may assume $\tilde{r} = \tilde{r}(1) < 1$. Denote $S_n = \sum_{k=1}^{n} \xi_k$, $\overline{X}_n = \frac{1}{n} S_n$. By Lemma 1 in Bradley [2], we have

$$\mathrm{E}\{(S_n)^2\} \le \frac{1+\tilde{r}}{1-\tilde{r}} \sum_{k=1}^{n} \mathrm{E}\{(\xi_k)^2\} . \tag{13}$$

This implies that $\overline{X}_n \to 0$ as $n \to \infty$ in L_2 and in probability. Indeed, $\mathrm{E}\{(\overline{X}_n)^2\} \le C(\frac{1}{n})^2 \sum_{k=1}^{n} \mathrm{E}\{(\xi_k)^2\} \to 0$ by (2). To prove the theorem, it suffices therefore to verify that $(\overline{X}_n)^2$ converges with probability 1. To this end, squaring the trivial recurrence $\overline{X}_{n+1} = n\overline{X}_n/(n+1) + \xi_{n+1}/(n+1)$ we get

$$\begin{aligned} (\overline{X}_{n+1})^2 - (\overline{X}_n)^2 &\le \left(\frac{1}{n+1} \right)^2 (\xi_{n+1})^2 + 2 \left(\frac{1}{n+1} \right)^2 |S_n||\xi_{n+1}| \\ &= A_n \quad \text{(say)}. \end{aligned} \tag{14}$$

From (2) we get

$$\sum_n \left(\frac{1}{n+1} \right)^2 \mathrm{E}\{(\xi_{n+1})^2\} < \infty ,$$

hence

$$\sum_n \left(\frac{1}{n+1} \right)^2 (\xi_{n+1})^2$$

converges a.s. Also it is easy to see that by the trivial inequality $2|ab| \le a^2 + b^2$ and (13)

$$\sum_n 2 \left(\frac{1}{n+1} \right)^2 \mathrm{E}\{|S_n||\xi_{n+1}|\} \le \sum_n n^{-3/2} \mathrm{E}\{(\xi_n)^2\} + \frac{1+\tilde{r}}{1-\tilde{r}} \sum_n n^{-5/2} \sum_{k=1}^{n} \mathrm{E}\{(\xi_k)^2\} .$$

Since

$$\sum_n n^{-5/2} \sum_{k=1}^{n} \mathrm{E}\{(\xi_k)^2\} \le C \sum_k k^{-3/2} \mathrm{E}\{(\xi_k)^2\} ,$$

thus by (2) the series $\sum_n A_n$ converges a.s. From (14) we have

$$(\overline{X}_{n+m})^2 - (\overline{X}_n)^2 \le \sum_{k=n}^{n+m} A_k .$$

Therefore $\limsup_{n\to\infty}(\overline{X}_n)^2 \leq \liminf_{n\to\infty}(\overline{X}_n)^2$ a.s., which concludes the proof.

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