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A Sensitivity Estimate for Boolean Functions

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Abstract—A Boolean response to a random binary input of length n can be modeled as a $\{0,1\}$-valued function v defined on a discrete probability space Ω of all subsets of a finite set of size n. An $\omega \in \Omega$ represents the locations of 1's in the input. For a particular j^{th} location, $1 \le j \le n$, we assume that 1 appears with probability ρ_j independently of other locations. Then, for $\bar\rho = (\rho_1, \dots, \rho_n)$, we will investigate $P_{\bar\rho}(v = 1)$ as a function of $\bar\rho$. Using the sharp version of the Khinchin inequality, we give an upper estimate for the ℓ_2 norm of the gradient of $P_{\bar\rho}(v = 1)$ evaluated at $\bar\rho = (1/2, \dots, 1/2)$ (cf. (5) below). For monotone functions, the estimate applies also to vector of influences of Boolean functions. We also provide a handy expansion of $P_{(\cdot)}(v = 1)$ based on a Fourier expansion of v (cf. (4) below).

Numerical analysis of the bounds leads to the conjecture about the sharp bound that depends on cardinality of the underlying set; the sharp version of the Khinchin inequality is also conjectured.

Keywords—Banzhaf index, Sensitivity, Boolean functions.

1. INTRODUCTION

We shall be interested in Boolean functions, i.e., $\{0,1\}$-valued functions on a discrete space Ω consisting of all subsets of $\{1, 2, \dots, n\}$. We treat Ω as a probability space and will assign the uniform probability $P(C) = 2^{-n}$, $C \in \Omega$. The expected value with respect to probability measure $P(\cdot)$ will be denoted by $E(\cdot)$. Boolean function $v(\cdot)$ defines a $\{0,1\}$-valued random variable on Ω and we shall assume that

$$p = P(v = 1) \tag{1}$$

is known. To simplify the notation, we assume $p \le 1/2$; in general, in our bounds p should be replaced by $p \wedge (1 - p)$.

Function v will be analyzed using the auxiliary stochastically independent random variables ϵ_j (representing flips of the j^{th} coin), defined by

$$\epsilon_j(C) = \begin{cases} -1 & \text{if } j \notin C, \\ 1 & \text{if } j \in C. \end{cases}$$

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We shall be interested in the coefficients

$$\beta(i) = E(\epsilon_i v). \tag{2}$$

The coefficients $\beta(j)$ are of interest in game theory and voting systems (Banzhaf index), see [1]. They also play a role in computer science in analyzing threshold functions (Chow index) and neural networks, see [2,3]. For monotone v, numbers β_j are the same as the so-called influences $\mathrm{Inf}_j(v)$, see [2,4]. The average sensitivity of v is then defined as $\sum_j \beta(j)$. In this language, our goal is to present a sharp estimate for the ℓ_2 norm of the influence vector, i.e., $\sum_{j=1}^n \beta^2(j)$. Notice that the gradient interpretation below points out that the sum of the squares might be a more natural global measure of sensitivity than the sum of influences.

Here is a short argument relating β_j to the rate of change of the probability $P(v = 1)$. This interpretation manifests itself when more general families of probability measures are introduced; then $\left(\sum_j \beta_j^2\right)^{1/2}$ quantifies how perturbations from uniform assignment of probability affect $P(v = 1)$. For $0 \le \rho_j \le 1$, consider a parametric probability measure $P_{\bar\rho}(\cdot)$ defined on the probability space Ω by

$$P_{\bar\rho}(C) = \prod_{j\in C} \rho_j \prod_{j\notin C} (1-\rho_j).$$

In this notation, the uniform $P(\cdot)$ defined previously equals $P_{(1/2,\dots,1/2)}(\cdot)$. It is easy to check either directly, or from (4) below, that

$$\frac{d}{d\rho_j} P_{\bar\rho}(v = 1)\Big|_{\rho_j=1/2} = 2\beta(j).$$

Average sensitivity $\sum_{j=1}^n \beta_j$ is given by a similar rate-of-change expression, when all $\rho_j = \rho$ are equal. The last result is actually related to Russo's formula in percolation theory, see [5, (2.25)], and it is also known in the context of multilinear extensions of games.

For $T \in \Omega$, denote $f_T = \prod_{j\in T} \epsilon_j$, with the convention $f_\emptyset = 1$. Then, $\{f_T(\cdot)\}$ is an orthonormal basis (the so-called Walsh system) of the finite-dimensional vector space $L_2(\Omega, P)$ of the square integrable random variables on Ω. In particular, we have the orthogonal (Fourier) expansion

$$v(\cdot) = \sum_{T\in\Omega} \alpha_T f_T(\cdot). \tag{3}$$

Notice that coefficients in (3) are $\alpha_\emptyset = E(v) = P(v = 1) = p$, and from (2), we have

$$\alpha_{\{i\}} = \beta(i).$$

Expansion (3) leads to the expansion for $P_{\bar\rho}(v = 1)$ by the following calculation. Writing $\rho_j = (1 + \delta_j)/2$, we have

$$P_{\bar\rho}(v = 1) = \sum_{\{C:\ v(C)=1\}} \prod_{j=1}^n \frac{1 + \epsilon_j(C)\delta_j}{2} = E\left\{ v \prod_{j=1}^n (1 + \epsilon_j \delta_j) \right\}.$$

Therefore, we obtain the (Taylor) expansion

$$P_{\bar\rho}(v = 1) = \sum_{T\in\Omega} \alpha_T \prod_{j\in T} (2\rho_j - 1). \tag{4}$$

Our main result is the following upper bound for the ℓ_2 norm of the vector $[\beta(1), \dots, \beta(n)]$. Notice that the right-hand side of inequality (5) doesn't allow for dependence in n; a generalization is mentioned in Remark 4 of Section 3. The bound is also valid for the sum of the squares of the Banzhaf Index in game theory, containing [6, Theorem 1] as a special case corresponding to $\theta = 1/2$.

THEOREM 1. *For any $\{0,1\}$-valued $v(\cdot)$ and $p \le 1/2$ defined by (1), we have*

$$\left(\sum_{i=1}^{n} \beta^2(i)\right)^{1/2} \le \frac{1}{\sqrt{2}} \inf_{0<\theta\le 1/2} \left(\frac{1}{\sqrt{\pi}}\Gamma\left(\frac{1+\theta}{2\theta}\right)\right)^{\theta} (2p)^{1-\theta}. \tag{5}$$

In particular, for small p choose $\theta = -1/\log p$. By Stirling's approximation, we get the following corollary.

COROLLARY 1. *As $p \to 0$,*

$$\sum_{i=1}^{n} \beta^2(i) \ll ep^2 \log \frac{1}{p},$$

(in the sense that the limsup *of the quotient is bounded by 1).*

This should be compared with the corresponding lower bounds given in [2, Theorem 3.1], see also [7, Theorem 0]. In particular, from [7, Theorem 0] for $p \le C2^{-n}$, we have $\sum_{j=1}^{n} \beta_j^2 \gg \left(\log 2 + \frac{C}{n}\right) p^2 \log 1/p$, showing that for small values of p and large n, inequality (5) is sharp up to a multiplicative factor.

Theorem 1 is valid (with minor modifications) in a more general setup, when v is not necessarily Boolean. In this context, we should point out that [7] considers influences on a more general product space. In a more general setup, it might be natural to extend the definition of the influence of any random variable X on a not necessarily $\{0,1\}$-valued v as the random variable $E(v \mid X)$. It is not clear, however, if a "rate of change" interpretation could then be found.

Our proof of Theorem 1 is based on the Khinchin inequality. The Khinchin inequality for more general families of orthogonal functions f_T and in another range of parameters (with $q < 2$ rather than $q > 2$) was used for lower bounds in [3]. For other Khinchin-like inequalities for subsets of the orthogonal functions $\{f_T\}$, see [8].

2. PROOF

From (3), we have

$$p^2 + \sum_{i=1}^{n} \beta^2(i) \le \sum_{T\in\Omega} \alpha_T^2 = E\left(v^2\right) = p, \tag{6}$$

giving

$$\sum_{i=1}^{n} \beta^2(i) \le p(1-p).$$

As it was pointed out in [6], this can be improved as follows. Consider $\tilde{v}(C) := v(C^c)$. Since $\epsilon_j(C^c) = -\epsilon_j(C)$, we have $E(\epsilon_j \tilde{v}) = -\beta(j)$. Therefore, for $V := v - \tilde{v}$, we have $E(\epsilon_j V) = 2\beta(j)$ and reasoning as in (6), we get

$$\sum_{i=1}^{n} \beta^2(i) \le \frac{1}{4} E\left(V^2\right).$$

Since for any $q \ge 1$,

$$E(|V|^q) = P(v=1, \tilde{v}=0) + P(v=0, \tilde{v}=1) \le 2(p \wedge (1-p)) = 2p, \tag{7}$$

we get

$$\sum_{i=1}^{n} \beta^2(i) \le \frac{1}{2} p. \tag{8}$$

To prove Theorem 1, we use the above symmetrization and two auxiliary results. The following is a sharp version of the Khinchin inequality, see [9, p. 265, Theorem B], see also Remark 1 in Section 3 below.

LEMMA 1. *For $2 \le q < \infty$ and any real coefficients $\{a_j\}$, we have*

$$\left(E\left|\sum_{i=1}^{n} a_i\epsilon_i\right|^q\right)^{1/q} \le \sqrt{2}\left(1/\sqrt{\pi}\,\Gamma\left(\frac{q+1}{2}\right)\right)^{1/q}\left(\sum_{i=1}^{n} a_i^2\right)^{1/2}. \tag{9}$$

For $1 \le q \le \infty$, consider the Banach space

$$\mathcal{L}_q = \{X \in L_q(\Omega) : E(X) = 0\}$$

with the inherited norm $||X||_{\mathcal{L}_q} = (E|X|^q)^{1/q}$, $q < \infty$. Clearly, $\mathcal{L}_q$ is isometric to the quotient of $L_q(\Omega)$ by the one-dimensional subspace generated by the constants; from the general theory, it is easy to check that the conjugate space $(\mathcal{L}_q)^*$ is isometric to $\mathcal{L}_{q'}$ where q' is the conjugate number, $1/q + 1/q' = 1$.

Consider now the linear operator

$$A : \mathcal{L}_{q_1} \to \mathcal{L}_{q_2}$$

given by

$$A(X) = \sum_{i=1}^{n} \epsilon_i E(\epsilon_i X).$$

Clearly, for $q_1 = q_2 = 2$, operator A is the orthogonal projection onto the $\mathrm{Span}\{f_{\{1\}}, \dots, f_{\{n\}}\}$; hence,

$$||AX||_{\mathcal{L}_2} \le ||X||_{\mathcal{L}_2}.$$

The relevance of this operator is obvious—for instance, inequality (8) can be rewritten as

$$\sum_{i=1}^{n} \beta^2(i) = \frac{1}{4}||AV||_2^2 \le \frac{1}{4}E\left(V^2\right). \tag{10}$$

LEMMA 2. *For $1 < q' \le 2$, we have*

$$||A||_{\mathcal{L}_{q'} \to \mathcal{L}_2} \le \sqrt{2}\left(\Gamma\frac{(q+1)/2}{\sqrt{\pi}}\right)^{1/q}, \tag{11}$$

where $1/q + 1/q' = 1$.

PROOF. Indeed, $q \ge 2$ and by (9), we have

$$||A||_{\mathcal{L}_2 \to \mathcal{L}_q} \le \sqrt{2}\left(\Gamma\frac{(q+1)/2}{\sqrt{\pi}}\right)^{1/q}.$$

Inequality (11) now follows from the fact that the adjoint operators have the same norm $||A|| = ||A^*||$ and from an easy observation that $A^* : \mathcal{L}_{q'} \to \mathcal{L}_2$ is given by the same formula as A. ∎

PROOF OF THEOREM 1. As before, let $V = v - \tilde{v}$. Then, by (10)

$$\sum_{i=1}^{n} \beta^2(i) = \frac{1}{4}||AV||_2^2.$$

From (11), we get for arbitrary $1 < q' \le 2$

$$||AV||_2 \le \frac{1}{\sqrt{2}}\left(\Gamma\frac{(2q'-1)/(2q'-2)}{\sqrt{\pi}}\right)^{(q'-1)/q'}||V||_{q'}.$$

Since by (7), we have $E(|V|^{q'}) \le 2p$, inequality (5) now follows by substituting $\theta = 1 - 1/q'$. ∎

3. CONCLUDING REMARKS

(1) For $q \geq 3$, [10] (see also [11]) gives the best constants in (9) for each $n \geq 1$. (Notice that [10] states the inequality for all $q \geq 2$; however, there is a minor error in the paper and the proof goes through only for $q \geq 3$.)

This allows further improvements in Theorem 3.1 giving an additional bound

$$\left(\sum_{i=1}^{n} \beta^2(i)\right)^{1/2} \leq \frac{1}{2} \inf_{0<\theta\leq 1/3} \frac{1}{\sqrt{n}} \left(\frac{1}{2^n}\sum_{k=0}^{n} |n-2k|^{1/\theta} \binom{n}{k}\right)^{\theta} (2p)^{1-\theta}. \tag{12}$$

(2) In the range $0 < \theta \leq 1/3$, the right-hand side of (5) is the limit of (12) as $n \to \infty$. Numerical analysis of the expression

$$\inf_{0<\theta\leq 1/2} \frac{1}{\sqrt{n}} \left(\frac{1}{2^n}\sum_{k=0}^{n} |n-2k|^{1/\theta} \binom{n}{k}\right)^{\theta} (2p)^{1-\theta} \tag{13}$$

indicates, however, that one cannot take the infimum in (12) over the whole interval $0 < \theta \leq 1/2$. This, in particular, shows that the result of [11] does not extend directly to exponents $q \geq 2$; the bound fails already for $n = 3$ and $q \approx 2.28$, giving estimates lower than the actual maxima in the Appendix.

(3) Further numerical evidence indicates that the inequality from [11] holds true in the range of exponents $2 < q < 3$ for all even n, while for n odd, the inequality we conjecture is

$$E\left|\sum_{j=1}^{n} a_j \epsilon_j\right|^q \leq (n-1)^{-q/2} E\left|\sum_{j=1}^{n-1} \epsilon_j\right|^q \left(\sum_{j=1}^{n} a_j^2\right)^{q/2}. \tag{14}$$

(4) Numerical evidence from the tables in the Appendix indicates that inequality (12) is more accurate for small values of p. Since the expression under the infimum in (12) is smaller than the one in (5), it is clear that the optimal value of θ exceeds $1/3$ for larger p.

The conjectured form of Khinchin inequality would imply the bound

$$\left(\sum_{j=1}^{n} \beta_j^2\right)^{1/2} \leq \inf_{0<\theta<1/2} \frac{1}{\sqrt{n+1}} \left(\frac{1}{2^{n+1}}\sum_{k=0}^{n+1} |n+1-2k|^{1/\theta} \binom{n+1}{k}\right)^{\theta} (2p)^{1-\theta} \tag{15}$$

for odd n. For more accuracy, one could actually switch between n and $n+1$ in appropriate ranges of θ; the choice of $n+1$ works for all θ when n is odd. Expression (13) is conjectured for even n. (Both fail outside their conjectured range, i.e., (15) fails for n even and (13) fails for n odd.)

APPENDIX
NUMERICAL COMPARISON

In this section, we present the numerical comparison of several bounds for the renormalized sum $4^n \sum_{j=1}^{n} \beta^2(j)$. Besides (5) and (12), we also analyze conjectured bounds and trivial inequalities (8) and

$$\sum_{j=1}^{n} \beta_j^2 \leq np^2. \tag{16}$$

Notice that the former corresponds to $\theta = 1/2$ and the latter corresponds to $\theta = 0$ in (12).

The rows labeled "Actual max" in the tables correspond to the maximal sum of the renormalized ℓ_2 norms over the $v(\cdot)$ corresponding to the so-called *simple games*; according to a trusted source, this is known to be the extreme case; then by [6, Proposition 1(ii)], it is enough to consider threshold functions only. Those were found by hand calculations. A computer program searching for extremal v by choosing random monotone v was then written and the largest value of the sum of squares found is reported below (all the extremals for $n = 3, 4$ were quickly recovered by the program).

The estimates were obtained by direct search through the discrete partition of the range of θ. The gamma function was approximated by its asymptotic expansion as given in [12]. The accuracy of both approximations is difficult to judge; for instance, the answers we got were quite sensitive to divisibility properties of the size of partition used. (We explain this by the fact that values $\theta = 1/3, 1/2$ are sometimes optimal—we settled on using partition of size 600.)

Table 1. Comparison of bounds for $4^n \sum \beta^2(j)$ for $n = 4$.

$2^n p$	1	2	3	4	5	6	7	8
Ineq. (8)	8.0	16.0	24.0	32.0	40.0	48.0	56.0	64.0
Ineq. (16)	4.0	16.0	36.0	64.0	100.0	144.0	196.0	256.0
Ineq. (5)	4.0	13.7	23.3	32.0	40.0	48.0	56.0	64.0
Ineq. (12)	4.0	12.6	22.7	33.3	44.8	57.1	70.2	83.9
Actual max	4	12	20	32	36	44	52	64
Conj. (13)	4.0	12.6	22.7	32.0	40.0	48.0	56.0	64.0

Table 2. Comparison of bounds for $4^n \sum \beta^2(j)$ for $n = 5$.

$2^n p$	2	4	6	8	10	12	14	16
Ineq. (8)	32.0	64.0	96.0	128.0	160.0	192.0	224.0	256.0
Ineq. (16)	20.0	80.0	180.0	320.0	500.0	720.0	980.0	1280.0
Ineq. (5)	19.6	54.7	93.3	128.0	160.0	192.0	224.0	256.0
Ineq. (12)	18.2	51.6	91.1	133.7	180.0	229.5	281.9	336.8
Largest found	16	48	80	128	144	176	208	256
Conj. (15)	17.5	52.2	91.8	128.0	160.0	192.0	224.0	256.0

Table 3. Comparison of bounds for $4^n \sum \beta^2(j)$ for $n = 6$.

$2^n p$	4	8	12	16	20	24	28	32
Ineq. (8)	128.0	256.0	384.0	512.0	640.0	768.0	896.0	1024.0
Ineq. (16)	96.0	384.0	864.0	1536.0	2400.0	3456.0	4704.0	6144.0
Ineq. (5)	78.4	218.7	373.2	512.0	640.0	768.0	896.0	1024.0
Ineq. (12)	73.9	208.9	367.8	539.8	726.8	926.9	1138.4	1360.2
Largest found	64	192	320	512	576	704	832	1024
Conj. (13)	69.9	208.9	367.1	512.0	640.0	768.0	896.0	1024.0

Table 4. Comparison of bounds for $4^n \sum \beta^2(j)$ for $n = 7$.

$2^n p$	8	16	24	32	40	48	56	64
Ineq. (8)	512.0	1024.0	1536.0	2048.0	2560.0	3072.0	3584.0	4096.0
Ineq. (16)	448.0	1792.0	4032.0	7168.0	11200.0	16128.0	21952.0	28672.0
Ineq. (5)	313.5	874.9	1492.8	2048.0	2560.0	3072.0	3584.0	4096.0
Ineq. (12)	298.2	841.9	1474.2	2163.5	2913.2	3714.8	4562.5	5451.6
Largest found	256	768	1280	2048	2304	2816	3328	4096
Conj. (15)	288.1	846.8	1475.2	2048.0	2560.0	3072.0	3584.0	4096.0

Table 5. Comparison of bounds for $4^n \sum \beta^2(j)$ for $n = 8$.

$2^n p$	16	32	48	64	80	96	112	128
Ineq. (8)	2048.0	4096.0	6144.0	8192.0	10240.0	12288.0	14336.0	16384.0
Ineq. (16)	2048.0	8192.0	18432.0	32768.0	51200.0	73728.0	100352.0	131072.0
Ineq. (5)	1254.1	3499.6	5971.3	8192.0	10240.0	12288.0	14336.0	16384.0
Ineq. (12)	1200.7	3387.1	5295.8	8696.3	11709.8	14932.2	18339.4	21913.3
Largest found	1024	3072	5632	8192	9216	9696	10360	16384
Conj. (13)	1152.2	3387.1	5900.8	8192.0	10240.0	12288.0	14336.0	16384.0

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