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A remark on the connection between the large deviation principle and the central limit theorem

Wlodzimierz Bryc
Department of Mathematics, University of Cincinnati, OH, USA

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Abstract: We point out that under a suitable regularity condition the central limit theorem can be obtained as a consequence of the large deviation principle.

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1. Results

Let $\{X_t\}_{t\geqslant 1}$ be a family of real random variables. An often useful step in establishing the large deviation principle for $\{X_t\}$ is to show that there is $\varepsilon > 0$ such that

$$\mathbb{L}(a) := \lim_{t\to\infty} t^{-1} \log E\{\exp(taX_t)\} \tag{1.1}$$

exists for all real a, $|a| < \varepsilon$; see e.g., Dembo and Zeitouni (1993, Theorem 2.3.6) or Cox and Griffeath (1984, Lemma 1).

In some more algebraic proofs of (1.1), however, one gets the convergence for complex values as well. For instance, the proofs in Burton and Dehling (1990), Cox and Griffeath (1984) and in Bryc and Smolenski (1993) admit complex arguments. The purpose of this short note is to point out that in such circumstances the Central Limit Theorem follows by a very simple argument. In Cox and Griffeath (1984) a similar approach based on real arguments is described and used; however, their paper depends on an unpublished theorem of R.S. Ellis. The only other relevant general result in print that we are aware of is Iagolnitzer and Souillard (1979), who deal with the averages of two valued random variables; their proof does not seem to extend to random variables with infinite support.

Proposition 1. *Suppose $\{X_t\}_{t\geqslant 1}$ satisfies $E\{X_t\} = \mathbf{0}$. If there is $\varepsilon > 0$ such that*

$$\mathbb{L}(z) = \lim_{t\to\infty} t^{-1} \log E\{\exp(tzX_t)\} \tag{1.2}$$

exists for all complex z, $|z| < \varepsilon$ then $t^{1/2}X_t$ converges in distribution to the N$(0, \sigma)$ *distribution, where* $\sigma^2 = \mathbb{L}''(0) \geqslant 0$.

Remark 1. The use of complex numbers in (1.2) can be avoided with the help of some additional assumptions, as in Cox and Griffeath (1984, Lemma 1); however, the assumption that all the functions under consideration are real-analytic is not sufficient. For instance, consider symmetric random variables $\{X_t\}_{t \geqslant 1}$ with distributions $P(|X_t| > x) = \exp(-x^2 t)$. The moment generating functions

$$E\{\exp(tyX_t)\} = 1 + \tfrac{1}{2} y t^{-1/2} \exp\left(\tfrac{1}{4} t y^2\right) \int_{y\sqrt{t}/2}^{y\sqrt{t}/2} \mathrm{e}^{-u^2}\, \mathrm{d}u$$

are analytic; their normalized logarithms are real-analytic and converge to (analytic) $\mathbb{L}(y) = \frac{1}{4} y^2$, but the convergence holds for the real arguments y only and the central limit theorem fails. On the other hand, the large deviation principle holds with the Gaussian rate function.

Proposition 1 can also be used to obtain the central limit theorem as a consequence of the large deviation principle. To state the result we need to introduce some notation. Let $Ze_t \subset \mathbb{C}$ denote the set of zeros of $z \to E\{\exp(tzX_t)\}$. Under integrability condition (1.3) below, each of the functions $a \to E\{\exp(tzX_t)\}$ is analytic in a neighborhood of 0, see e.g. Lukacs (1970); hence each set Ze_t consists of the isolated points of the complex domain.

Proposition 2. *If*

$$\sup_t \left(E\{\exp(t\varepsilon |X_t|)\}\right)^{1/t} < \infty \quad \textit{for some } \varepsilon > 0, \tag{1.3}$$

and

$$0 \notin \operatorname{closure}\left(\bigcup_{t \geqslant 1} Ze_t\right) \tag{1.4}$$

then the large deviation principle implies (1.2).

Remark 2. Condition (1.4) is usually difficult to verify directly, but follows from (1.2); it is sometimes considered in statistical physics literature, where $E\{\exp(tzX_t)\}$ is called a partition function. Once (1.4) is satisfied, the rate function in the large deviation principle is strictly convex and in addition to the CLT we also get exponential convergence, see Ellis (1985, Theorem II.6.3).

Remark 3. In general, the central limit theorem is not a consequence of the large deviation principle, see the example given in Remark 1.

2. Proofs

Proof of Proposition 2. By (1.4), there is $\varepsilon > 0$ such that functions $L_t : \mathbb{C} \to \mathbb{C}$ defined by

$$L_t(z) = t^{-1} \log E\{\exp(tzX_t)\}$$

are analytic in the disk $|z| < 3\varepsilon$. By Carathéodory's inequality, see e.g. Levine (1956, p. 17, Theorem 8), we have

$$|L_t(z)| \leqslant 2t^{-1} \log \sup_{|z| \leqslant 2\varepsilon} |E\{\exp(tzX_t)\}| \leqslant 2t^{-1} \log E\{\exp(t\varepsilon |X_t|)\}$$

for $|z| < \varepsilon$. Therefore, by (1.3),

$$\sup_{t \geqslant 1, |z| \leqslant \varepsilon} |L_t(z)| < \infty.$$

Furthermore, by the large deviation principle and (1.3), $\{L_t(x)\}_{t\geqslant 1}$ converges for real x. Therefore by induced convergence, see e.g. Hille (1962, Theorem 15.3.4), the limit (1.2) exists for each $|z|\leqslant\varepsilon$; moreover, convergence is uniform in the disc $|z|\leqslant\varepsilon$ and the limit is an analytic function of the argument z. □

Proof of Proposition 1. This proof is a continuation of the previous proof. Since (1.2) implies (1.4), by the previous argument, convergence in (1.2) is uniform in a disc. Thus there is $\varepsilon>0$ such that for $k=1,2,\ldots,$

$$(k!)^{-1}2\pi\mathrm{i}\frac{\partial^k}{\partial z^k}\mathbb{L}(z)\bigg|_{z=0}=\int_{|z|=\varepsilon}\mathbb{L}(z)/z^{k+1}\,\mathrm{d}z=\lim_{t\to\infty}t^{-1}\int_{|z|=\varepsilon}\log E\{\exp(tzX_t)\}/z^{k+1}\,\mathrm{d}z. \tag{2.1}$$

The value of the integral $\int_{|z|=\varepsilon}\log E\{\exp(tzX_t)\}/z^{k+1}\,\mathrm{d}z$ is not affected by changing the integration path from $|z|=\varepsilon$ to $|z|=\varepsilon/t^{1/2}$. Therefore substituting $u=zt^{1/2}$ we get

$$\lim_{t\to\infty}t^{k/2-1}\int_{|u|=\varepsilon}\log E\{\exp(t^{1/2}uX_t)\}/u^{k+1}\,\mathrm{d}u=(k!)^{-1}2\pi\mathrm{i}[\partial/\partial z]^k\mathbb{L}(z)|_{z=0}. \tag{2.2}$$

Since

$$\int_{|u|=\varepsilon}\log E\{\exp(t^{1/2}uX_t)\}/u^{k+1}\,\mathrm{d}u=(k!)^{-1}2\pi\mathrm{i}[\partial/\partial x]^k\log E\{\exp(t^{1/2}xX_t)\}|_{x=0}$$

equals $(k!)^{-1}2\pi\mathrm{i}$ times the kth cumulant of real random variable $t^{1/2}X_t$, from (2.2) we see that for $k\geqslant 3$ the kth cumulant of $t^{1/2}X_t$ converges to zero as $t\to\infty$; moreover,

$$[\partial/\partial x]^k\log E\{\exp(t^{1/2}xX_t)\}|_{x=0}=\mathrm{O}(t^{1-k/2})\quad\text{as } t\to\infty.$$

For $k=2$, (2.2) reads

$$\operatorname{Var}[t^{1/2}X_t]\to[\partial/\partial x]^2\mathbb{L}(x)|_{x=0}.$$

Since $E\{X_t\}=0$, all the cumulants converge to the corresponding cumulants of the normal distribution. This shows that $t^{1/2}X_t$ has asymptotically normal distribution with the variance $[\partial/\partial x]^2\mathbb{L}(x)|_{x=0}$. (For a more subtle normal convergence criterion, see Janson (1988).) □

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