PATTERNED RANDOM MATRICES

ARUP BOSE

Indian Statistical Institute
Kolkata

November 15, 2011

Introduction

- Patterned random matrices.
- Limiting spectral distribution.
- Extensions (joint convergence, free limit, half independence).
- Some questions.

Examples I

Sample variance covariance matrix (S matrix). $n^{-1}X_{n\times p}X^T_{p\times n}$, X is an IID matrix.

Wigner matrix $n^{-1/2}X_{n\times n}$ symmetric with IID entries.

Triangular symmetric Wigner matrix

$$W_n^u = \begin{bmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1(n-1)} & x_{1n} \\ x_{12} & x_{22} & x_{23} & \cdots & x_{2(n-1)} & 0 \\ & & & \vdots & & \\ x_{1(n-1)} & x_{2(n-1)} & 0 & \cdots & 0 & 0 \\ x_{1n} & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}, \quad (1)$$

$n^{-1}X_{n\times p}X^T_{p\times n}$ matrices, X is not necessarily IID.

Toeplitz matrix $((n^{-1/2}x_{|i-j|}))$.

Hankel matrix $((n^{-1/2}x_{i+j}))$.

Reverse Circulant matrix $((n^{-1/2}x_{i+j \ mod \ n}))$.

Examples II

$n^{-1/2}$IID matrix

Sample autocovariance matrix $((\gamma_{|i-j|}))$,

$$\gamma(k) = n^{-1} \sum_{j=1}^{n-k} x_j x_{k+j}.$$

Circulant, k circulant and its variants.

Band matrices. r–diagonal matrices.

Heavy tailed or dependent entries.

We shall focus on real symmetric matrices

Empirical spectral distribution (ESD)

ESD:

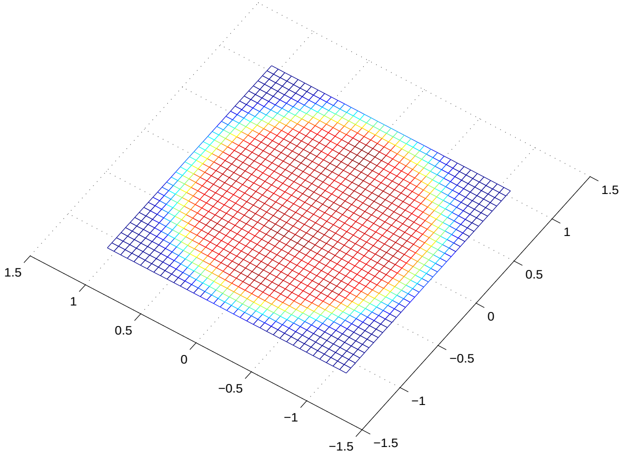
$$F_n(x) = n^{-1} \sum_{i=1}^{n} I\{\lambda_i \leq x\}.$$

Random probability distribution with mass $1/n$ *at each* λ_i.

How does it behave as $n \to \infty$?

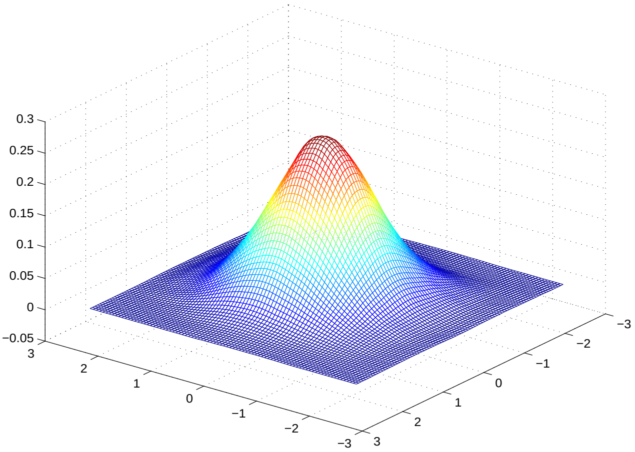
Limiting spectral distribution (LSD): the weak limit of ESD either almost surely or in probability.

IID



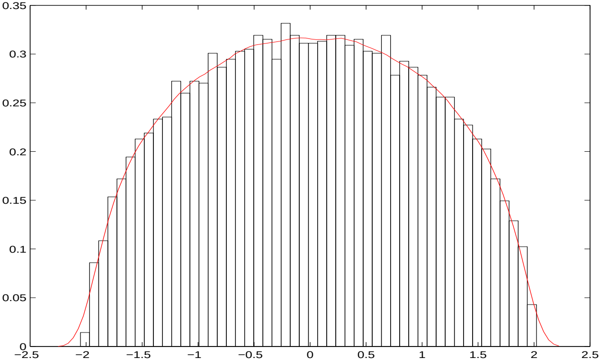
Scaled iid matrix of order 400 with Bernoulli entries.

Circulant



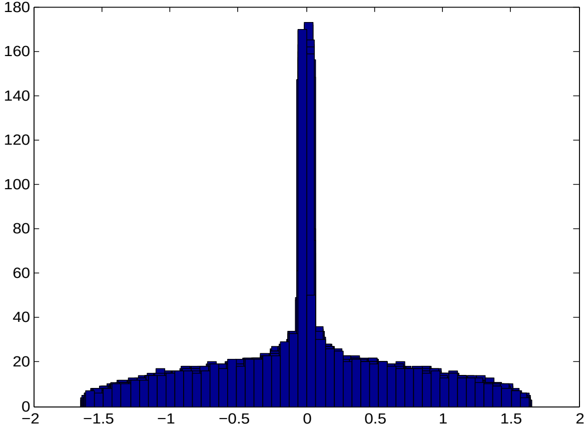
Scaled Circulant matrix of order 400 with Bernoulli entries.

Wigner



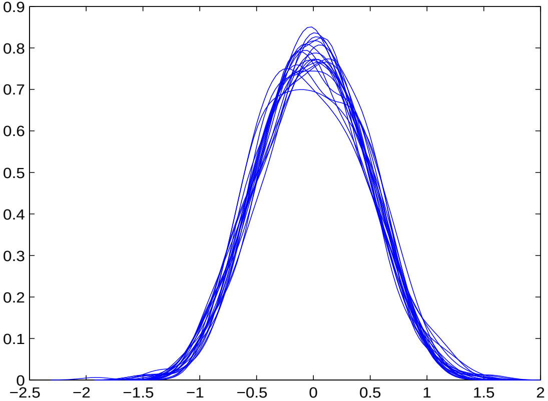
15 (scaled) Wigner matrices of order 400 with Bernoulli entries.

Triangular Wigner



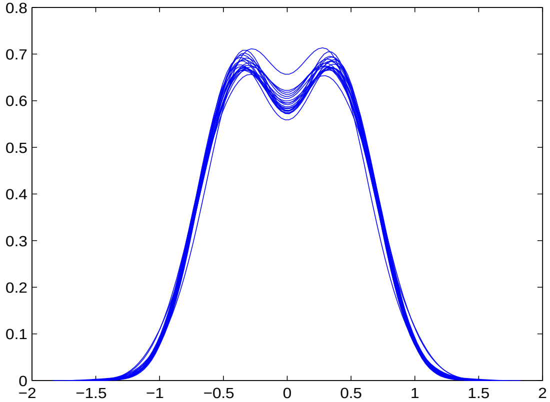
Triangular Wigner.

Toeplitz



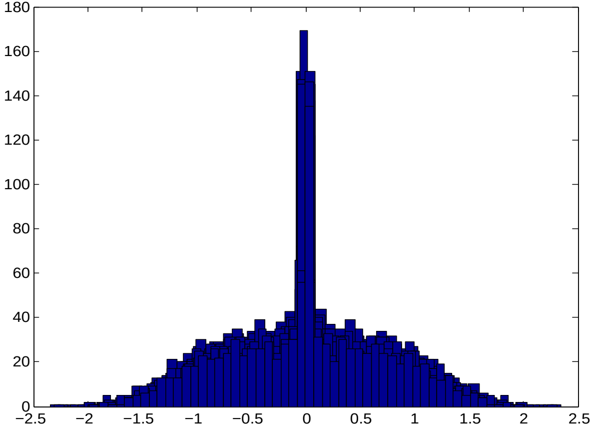
15 scaled Toeplitz matrix of order 400 with Bernoulli entries.

Hankel



15 scaled Hankel matrix of order 400 with Bernoulli entries.

Triangular Hankel



Triangular Hankel.

Input, link and patterned matrices

Input sequence: the sequence of variables used to construct the matrix: independent with mean zero and variance one.

For simplicity assume that it is uniformly bounded.

Patterned matrix:

$$P_n = ((x_{L(i,j)}))\ \ 1 \leq i,j \leq n.$$

Link function: L. Symmetry is assumed wherever applicable.

Interested in LSD of $n^{-1/2}P_n$.

Link functions of the matrices

Wigner: $L(i,j) = (\min(i,j), \max(i,j))$.

Toeplitz: $L(i,j) = |i-j|$.

Hankel: $L(i,j) = i+j$.

Reverse Circulant: $L(i,j) = (i+j) \mod n$.

Symmetric Circulant: $L(i,j) = n/2 - |n/2 - |i-j||$.

S: $L1(i,j) = (i,j)$, $L2(i,j) = (j,i)$.

Digression to CLT: pair partition

$\{x_i\}$ are independent uniformly bounded with mean zero and variance one. Let

$$Y_n = n^{-1/2}(x_1 + x_2 + \ldots + x_n).$$

Hence

$$\mathrm{E}[Y_n]^h = \frac{1}{n^{h/2}} \sum_{1 \le i_1, i_2, \ldots, i_h \le n} \mathrm{E}[x_{i_1} x_{i_2} \cdots x_{i_h}].$$

Taking expectation and using elementary order calculations,

$$\begin{aligned} \mathrm{E}[Y_n]^{2k+1} &= o(1) \\ \mathrm{E}[Y_n]^{2k} &= \frac{(2k)!}{2^k k!} + o(1). \end{aligned}$$

$\frac{(2k)!}{2^k k!}$ = Total number of *pair partitions* of $\{1, 2, \ldots 2k\}$.

$$(ab....a..b....)$$

.

Moment Method

The h-th moment of the ESD has the following nice form:

$$\frac{1}{n}\sum_{i=1}^{n}\lambda_i^h = \frac{1}{n}\mathrm{tr}(A^h) = \beta_h(A) \text{ (say)}.$$

If

(M1) $E[\beta_h(A_n)] \longrightarrow \beta_h$ (convergence of the average ESD).

(M2) $V[\beta_h(A_n)] \longrightarrow 0$.

(C) $\{\beta_h\}$ satisfies *Carleman's condition*:

$$\sum_{h=1}^{\infty}\beta_{2h}^{-1/2h} = \infty.$$

Then LSD of $\{A_n\}$ exists (in probability).

We will focus only on condition (M1).

Property B: subsequential limits, Gaussian domination, Carleman condition

Property B: Δ, the maximum number of times any variable appears in any row remains finite as dimension increases.

$\Delta = 1$: Wigner and Reverse Circulant.
$\Delta = 2$: Toeplitz, Hankel and Symmetric Circulant.

Result 1

Suppose Property B holds. Then the ESD of $\{n^{-1/2}P_n\}$ is tight a.s.. Any subsequential limit G satisfies,

(i) $\beta_{2k}(G) \leq \frac{(2k)!\Delta^k}{k!2^k}$ (implies Carleman's condition).

(ii) $\beta_{2k+1}(G) = 0$ (implies symmetry of G)

(iii) LSD exists for $\{n^{-1/2}P_n\}$ iff for every h,

$$\lim \mathrm{E}[\beta_h(n^{-1/2}P_n)] = \beta_h \text{ (say)}. \tag{2}$$

Circuit

h-th moment of $n^{-1/2}P_n$ is:

$$\frac{1}{n}\operatorname{Tr}\left(\frac{P_n}{\sqrt{n}}\right)^h = \frac{1}{n^{1+h/2}} \sum_{1\le i_1,i_2,\ldots,i_h\le n} x_{L(i_1,i_2)}x_{L(i_2,i_3)}\cdots x_{L(i_{h-1},i_h)}x_{L(i_h,i_1)}. \tag{3}$$

Circuit $\pi : \{0,1,2,\cdots,h\} \to \{1,2,\cdots,n\}$ with $\pi(0)=\pi(h)$.

Condition (M1):

$\mathrm{E}[\beta_h(n^{-1/2}P_n)] = \frac{1}{n^{1+h/2}}\sum_{\pi:\ \pi \text{ circuit}} \mathrm{E}\,\mathbb{X}_\pi \to \beta_h$ where

$$\mathbb{X}_\pi = x_{L(\pi(0),\pi(1))}x_{L(\pi(1),\pi(2))}\cdots x_{L(\pi(h-2),\pi(h-1))}x_{L(\pi(h-1),\pi(h))}.$$

Matched and pair matched circuits

Matched circuit: For any i, there is at least one $j \neq i$ such that

$$L(\pi(i-1), \pi(i)) = L(\pi(j-1), \pi(j)).$$

Pair matched circuit: such values occur only in pairs.

π *nonmatched* implies $\mathrm{E}(\mathbb{X}_\pi) = 0$. Hence

$$\mathrm{E}[\beta_h(n^{-1/2}P_n)] = \frac{1}{n^{1+h/2}} \sum_{\pi:\ \pi \text{ matched circuit}} \mathrm{E}\,\mathbb{X}_\pi.$$

Only matched circuits need to be considered.

Equivalence of circuits: words

π_1 and π_2 are *equivalent* if they match at the same positions.

Equivalence class $\leftrightarrow$ partition of $\{1, 2, \cdots, h\} \leftrightarrow$ *word* w of length h of letters where the first occurrence of each letter is in alphabetical order.

Example: $h = 5$. $\{\{1, 3, 5\}, \{2, 4\}\} \leftrightarrow$ *ababa*.

$\Pi(w)$ = Equivalence class corresponding to w.

(M1) can be written as:

$$\mathrm{E}[\frac{1}{n}\operatorname{Tr}\left(\frac{P_n}{\sqrt{n}}\right)^h] = \frac{1}{n^{1+h/2}}\sum_{w}\sum_{\pi\in\Pi(w)}\mathrm{E}\,\mathbb{X}_\pi \to \beta_h.$$

Limit moments: pair matched words

- The first sum is a finite sum.
- $\mathrm{E}\,\mathbb{X}_\pi = 1$ for pair matched words.

Let (for w of length h),

$$p(w) = \lim_n \frac{1}{n^{1+h/2}} \#\Pi(w) \quad \textit{whenever the limit exists.}$$

- Only pair matched words survive if we assume Property B:

$$p(w) = 0 \text{ if } w \text{ is not pair matched.}$$

Hence $\beta_{2k+1} = 0$ for all k.

IF $p(w)$ exists for each pair matched word, then

$$\beta_{2k} = \sum_{w \text{ pair matched}} p(w)$$

(w is of length $2k$).

The five matrices

Result 2

LSD exists for Wigner, Toeplitz, Hankel, Symmetric Circulant and Reverse Circulant.

Symmetric Circulant, Gaussian limit

- Recall that Guassian is an "upper bound".

- For Symmetric Circulant, $p(w) = 1$ for all (pair matched) w.
Hence the LSD is standard Gaussian.

- Not so for other link functions.

- Guassian limit is an "exception".

Symmetric words and Reverse Circulant

Symmetric word: each letter appears exactly once in an odd and exactly once in an even position.

abcabc is symmetric while *abab* is not.

There are $k!$ symmetric words of length $2k$.

For Reverse Circulant, $p(w) = 1$ for all symmetric words (and 0 otherwise). Hence the LSD is

$$f_R(x) = |x|e^{-x^2}, \; -\infty < x < \infty.$$

This is the distribution of the symmetrized square root of chi-square two distribution.

Catalan words, Wigner matrix and Semi Circle law

Catalan: a symmetric word where sequentially deleting all double letters leads to the empty word.

abcabc is symmetric but not Catalan. *abccba* is Catalan.

$$\text{Catalan words} \\ \Updownarrow \\ \text{noncrossing pair partitions} \\ \Updownarrow \\ \text{SRW paths, origin to origin, on or above the axis.}$$

There are $\frac{(2k)!}{k!(k+1)!}$ Catalan words of length $2k$.

For the Wigner matrix $p(w) = 1$ if w is Catalan (and 0 otherwise). Hence the LSD is the semicircle law

$$f_w(x) = \frac{1}{2\pi}\sqrt{4 - x^2}, \ -2 \le x \le 2.$$

Toeplitz and Hankel

For both, $p(w) = 1$ if w is Catalan.

For symmetric words $p_T(w) = p_H(w)$.

LSD are unbounded.

Hankel LSD is not unimodal.

Toeplitz LSD has a density which is bounded.

Table: Pair matched words and moments, X symmetric. Similar table for XX'

MATRIX	w Cat.	w sym. not cat.	Other w	β_{2k} or LSD
SC	1	1	1	$\frac{(2k)!}{2^k k!}$, $N(0,1)$
R	1	1	0	$k!$, $\pm\sqrt{\chi_2^2}$
T	1	$p_T(w) \neq 0,1$	$p_T(w) \neq 0,1$	$\frac{(2k)!}{k!(k+1)!} \leq \beta_{2k} \leq \frac{(2k)!}{k!2^k}$
H	1	$p_H(w) = p_T(w)$	0	$\frac{(2k)!}{k!(k+1)!} \leq \beta_{2k} \leq k!$
W	1	0	0	$\frac{(2k)!}{k!(k+1)!}$, semicircle

Triangular matrices

Result 3

LSD exists for triangular versions of Wigner, Toeplitz, Hankel, Symmetric Circulant and Reverse Circulant. The LSD is symmetric.

Triangular Wigner

Only Catalan words contribute but not equally.

LSD is supported in $[-\sqrt{e}, \sqrt{e}]$ with density unbounded at 0.

$$\beta_{2k} = \frac{k^k}{(k+1)!}.$$

Word	$p_u(w)$
aa	1/2
aabb	1/3
abba	1/3
aabbcc	1/4
abbcca	1/4
abbacc	5/24
aabccb	5/24
abccba	5/24

Catalan words and semicircle law

Let $\alpha_n =$ max occurrence of a variable in the matrix.

$\alpha_n = 2$: Wigner.

$\alpha_n = O(n)$: Symm. circ., Reverse Circ., Toeplitz and Hankel.

Result 4

(i) If $\alpha_n = o(n)$, then w non-Catalan implies $p(w) = 0$.

Any subsequential limit has support contained in $[-2\sqrt{\Delta}, 2\sqrt{\Delta}]$.

(ii) If $\Delta = 1$, then w Catalan implies $p(w) = 1$.

If further, $\alpha_n = o(n)$, then the LSD is semicircular.

Properties B and M, Catalan words and semicircle law

Property M: the maximum number of rows where matches between any two columns occur is finite across all dimensions.

This is satisfied for all the matrices we considered.

Result 5

(i) Suppose L satisfies Property B.

Then, $\beta_{2k} \geq \dfrac{(2k)!}{k!(k+1)!}$ for any subsequential limit.

(ii) Suppose L satisfies Properties B and M.

Then for any Catalan word w, $p(w) = 1$.

(iii) Suppose in (ii) we also assume $\alpha_n = o(n)$.

Then the LSD of $\{n^{-1/2}P_n\}$ is the semicircle law.

Joint convergence: Non comm. prob. space

Noncommutative probability space: $(\mathcal{A}, \phi)$. $\mathcal{A}$ is a unital algebra, $\phi : \mathcal{A} \to \mathbb{C}$ is a linear functional satisfying $\phi(1) = 1$.

$\mathcal{A}_n$ = $n \times n$ real symm. random matrices, $\phi_n = \frac{1}{n}[\mathrm{Tr}(\cdot)]$.

$\mathcal{A}$ = algebra of polynomials in J noncommutative variables $a_1, \ldots, a_J$.

Result 6

Consider J (indices) sequences of independent matrices $\{A_{i,n}, 1 \leq i \leq J\}$ with link function L satisfying Property B. If $p(w)$ exists, then for any k and any monomial,

$$\lim \mathrm{Tr}\left[\frac{1}{n}\frac{A_{i_1,n}}{\sqrt{n}} \cdots \frac{A_{i_k,n}}{\sqrt{n}}\right] = \lim \phi_n(a_{i_1} \ldots a_{i_k}) = \phi(a_{i_1} \ldots a_{i_k}) \text{ (say)}$$

exists almost surely. $(\mathcal{A}_n, \phi_n) \to (\mathcal{A}, \phi)$.

Joint convergence: total, free and half independence

• Symmetric Circulants commute.
Each index pair matched word contributes 1: totally indept.

• The Wigner limit.
Each indexed Catalan word contributes 1: free independent.

• The Reverse Circulants satisfy $ABC = CBA$ (half commute).
Each indexed symmetric word contributes 1: half independent.

Colors and indices

Different matrices: colors. So we have five colors (W, T, H, Symm. circ., Rev. circ.).

Multiple copies: indices.

Result 7

Joint convergence holds if we have several indices of all the five colors. The Wigner components are free of the rest of the four colors.

Half independence

Symmetric elements. $\{a_i\}_{i\in J} \subset \mathcal{A}$. For any $\{i_j\} \subset J$, let $a = a_{i_1}a_{i_2}\cdots a_{i_k}$. Let $E_i(a)$ and $O_i(a)$ = the number of times a_i has occurred in the even/odd positions in a.
a is symmetric (wrt $\{a_i\}_{i\in J}$) if $E_i(a) = O_i(a)$ for all $i \in J$.

Half independent elements. Suppose $\{a_i\}_{i\in J}$ half commute (that is $a_ia_ja_k = a_ka_ja_i$ for all i, j, k). $\{a_i\}$ are half independent if the following conditions are satisfied.

1. $\{a_i^2\}_{i\in J}$ are independent (joint moment splits).
2. For all nonsymmetric a, $\phi(a) = 0$.

Remarks

(i) Property B implies convergence of the moments $\leftrightarrow$ existence of LSD. Subsequential limits always exist. What further restrictions on the link function implies the convergence of moments?

(ii) Under what conditions on the link does the LSD have bounded or unbounded support? Density? One mode?

(iii) Classes of limits possible under Property B (?)

(iv) More on classes of contributing words and link functions?

(v) More about the LSD of T, H ?

(vi) More about the LSD of triangular T, H, ...?

(v) Further properties of joint convergence?

Collab.

RIDDHIPRATIM BASU, Univ. of California, Berkeley (Ph.D student)

SAYAN BANNERJEE, Univ. of Washington (Ph.D student)

ANIRBAN BASAK, Stanford University (Ph.D. student)

SOURAV CHATTERJEE, Courant Institute

SREELA GANGOPADHYAY, C.R. Rao Institute, Hyderabad

SHIRSHENDU GANGULY, Univ. of Washington (Ph.D student)

RAJAT SUBHRA HAZRA, University of Zurich

JOYDIP MITRA, Management Development Institute, Gurgaon

KOUSHIK SAHA, Bidhannagar College, Kolkata

ARNAB SEN, Cambridge University.

SANCHAYAN SEN, Courant Institute (Ph.D. student)

Selected references

Anderson, Guionnet and Zeitouni, O. (2009). *An Introduction to Random Matrices*. Cambridge University Press.

Bai (1999). Methodologies in spectral analysis of large dimensional random matrices, a review. Statistica Sinica

Bai and Silverstein (2006). *Spectral Analysis of Large Dimensional Random Matrices.* Science Press, Beijing.

Basu; Bose; Ganguly and Hazra (2011). Joint convergence of several copies of different patterned random matrices. Technical Report.

Bose and Banerjee (2010). Noncrossing partitions, Catalan words and the semicircle law. **J. Theoret. Probab**.

Bose; Gangopadhyay and Sen (2010). Limiting spectral distribution of XX' matrices. **Annales de l'Institut Henri Poincaré - Probabilités et Statistiques**.

Bose; Hazra and Saha (2010). Patterned Random Matrices and Method of Moments. **Proceedings of International Congress of Mathematicians, 2010.**

Bose; Hazra and Saha (2011a). Half independence and half cumulants. **Electronic Communications in Probability**.

Bose; Hazra and Saha (2011b). Convergence of joint moments for independent random patterned matrices. **Annals of Probability**.

Bose and Sen (2008). Another look at the moment method for large dimensional random matrices. **Elec. J. Probab.**.

Bryc; Dembo and Jiang (2006). Spectral measure of large random Hankel, Markov and Toeplitz matrices. **Ann. Probab.**

Mehta (1967). *Random matrices and the statistical theory of energy levels*. Academic Press.

Sen and Virag (2011). Absolute continuity of the limiting eigenvalue distribution of the random Toeplitz matrix. Preprint.

Tao; Vu and Krishnapur (2009). Random Matrices: universality of the ESDs and the circular Law. **Ann. Probab.**

Boundedness assumption

$$
\begin{aligned}
k_n &= \#\{L_k(i,j) : 1 \le i,j,k \le n\}, \\
\alpha_n &= \max_k \#\{(i,j) : L_n(i,j) = k,\ 1 \le i,\ j \le n\}.
\end{aligned}
$$

$\alpha_n = 2$ for Wigner and $\alpha_n = O(n)$ for Symmetric Circulant, Reverse Circulant, Toeplitz and Hankel.

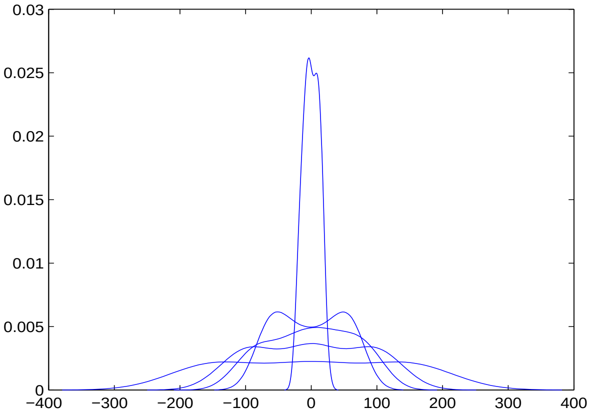
$k_n = O(n^2)$ for Wigner and $k_n = O(n)$ for Symmetric Circulant, Reverse Circulant, Toeplitz and Hankel.

Result

(The i.i.d. case): if $\alpha_n k_n = O(n^2)$, $k_n \to \infty$, then it is enough to restrict to bounded inputs among finite variance inputs.

Alternate assumption: (The non i.i.d. case) Any power of the input sequence is uniformly integrable.

Toeplitz matrix simulation with Cauchy entries



5 replications of the Toeplitz matrix of order 400: Cauchy entries.

Table: LSD of $S = \frac{XX'}{n}$ matrices.

MATRIX	w Cat.	w sym. not cat.	Other w	β_k or LSD
$p/n \to 0$ $\sqrt{\frac{n}{p}}(S - I_p)$, $\sqrt{\frac{n}{p}}(XX' - I_p)$, $X = T,\ H,\ R,\ C$	1 (in p)	0	0	$\frac{(2k)!}{k!(k+1)!}$, semicircle $\mathcal{L}_T$
$p/n \to y \neq 0, \infty$ S $X = T, H, R, C$	1	0	0	$\sum_{t=0}^{k-1} \frac{1}{t+1} \binom{k}{t} \binom{k-1}{t} y^t$ different, but universal