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Large Deviations for Quadratic Functionals of Gaussian Processes[1]

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The Large Deviation Principle (LDP) is derived for several quadratic additive functionals of centered stationary Gaussian processes. For example, the rate function corresponding to $1/T \int_0^T X_t^2 \, dt$ is the Fenchel-Legendre transform of $L(y) = -(1/4\pi) \int_{-\infty}^{\infty} \log(1 - 4\pi y f(s)) \, ds$, where X_t is a continuous time process with the bounded spectral density $f(s)$. This spectral density condition is strictly weaker than the one necessary for the LDP to hold for all bounded continuous functionals. Similar results are obtained for the energy of multivariate discrete-time Gaussian processes and in the regime of moderate deviations, the latter yielding the corresponding Central Limit Theorems.

KEY WORDS: Large deviations; moderate deviations; quadratic additive functionals; Gaussian processes.

1. INTRODUCTION

Recall that a collection $\{\mathbf{Z}_n\}$ of E-valued random variables satisfies the Large Deviation Principle (LDP) with speed $a_n \to 0$ and rate function $I: E \to [0, \infty]$, if the level sets $I^{-1}([0, b])$ are compact for all $b < \infty$, and

$$\liminf_{n \to \infty} a_n \log P(\mathbf{Z}_n \in A) \geqslant -\inf_{x \in A} I(x)$$

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for all open subsets $A \subset E$, while

$$\limsup_{n \to \infty} a_n \log P(\mathbf{Z}_n \in A) \leqslant - \inf_{x \in A} I(x)$$

for all closed subsets $A \subset E$.

Throughout most of the paper $E = \mathbb{R}$, except in Proposition 2, where $E = \mathbb{R}^2$, and in Proposition 3, where $E = \mathbb{R}^{d+1}$. Unless explicitly stated otherwise, LDPs in this paper are of speed $a_n = n^{-1}$ and we reserve n for discrete indices $n = 1, 2,...$ using T when working with continuous indices.

For a sample of results and references about the LDP for empirical measures of stationary processes under some restriction on the dependence see Refs. 6 and 12 [Sections 6.4 and 6.6].

The LDP for the empirical measures of stationary Gaussian processes is given in Ref. 15, see also Ref. 22 for the extension to Gaussian fields and Ref. 14 for an interesting case. The LDP for empirical means of bounded additive functionals of stationary Gaussian processes is a direct consequence of the results of Ref. 15. By approximation it applies also to functionals of the form $|X|^p$, $p < 2$.

In this paper, we study the LDP for quadratic functionals of stationary centered Gaussian processes that posses spectral density. These functionals receive the most attention in applications—for electrical engineering motivation, see Ref. 9; motivation from control theory, see Ref. 8; for statistical motivation, see Ref. 11.

Our main results, Theorems 1 and 2, can not be derived from the LDP,[15] since the moment generating function of square of a Gaussian variable is not finite everywhere. The condition of bounded spectral density in Theorems 1 and 2 is actually *strictly* weaker than the conditions of Ref. 15. See Ref. 5 for examples of processes for which these theorems apply while the LDP of Ref. 15 fails.

The Gärtner-Ellis theorem is also not suitable for deriving these theorems. Indeed, the mere existence of the relevant asymptotic logarithmic moment generating function at the boundary of its effective domain is not clear, and moreover, typically this function is not steep. We circumvent both problems by applying the parameter-dependent change of measure.[13] The delicacy of this issue is best demonstrated by Example 1, where the limiting logarithmic moment generating function of an additive $\mathbb{R}^2$-valued quadratic functional of a discrete time Gauss-Markov process is shown to violate the "prediction" of the Grenander-Szegö theory (i.e., (2.4) and (2.10)). Incidently, this case is not covered by the existing literature of LDPs for additive functionals of Markov chains.

Previous works overlapping our LDP results have either ignored these difficulties or added unnecessary restrictive conditions to mitigate them.

For example, Coursol and Dacunha-Vastelle[11] deals with quadratic forms in an implicit way and using the Grenander-Szegö method obtains a version of the LDP restricted to certain sets without explicit expressions for the rate function; Benitz and Bucklew[2] presents the heuristic reasoning motivating and facilitating much of our work but without rigorous proofs; in Ref. 10, the LDP of our Corollary 1 is stated under an additional technical assumption; in Ref. 8, explicit rate function is found for the special case of autoregressive AR(1) processes.

There exist many other works on quadratic forms of Gaussian random variables. For example, an early paper[20] uses the saddle point method to approximate the distribution for a fixed number of variables (see also Ref. 19), while for the Central Limit Theorem (CLT), see Refs. 1, 17, and 23 and the references therein. Analyzing moderate deviations of quadratic functions of Gaussian processes in Theorem 3, we recover some of these CLTs as well as a few ones.

Our results are stated in the next section, with proofs provided in Section 3.

2. RESULTS

The content of this section is as follows. For a continuous time process X_t with bounded spectral density $f(s)$, we show in Theorem 1 that $T^{-1}\int_0^T X_t^2\,dt$ satisfies the LDP with the rate function which is the Fenchel-Legendre transform of $L(y) = -1/4\pi \int_{-\infty}^{\infty} \log(1-4\pi y f(s))\,ds$. Theorem 2 provides the corresponding multivariate discrete-time result. For more moderate deviations, Theorem 3 states LDPs with speed a_n such that $na_n \to \infty$. The latter LDPs holds also for processes of unbounded spectral densities with Proposition 1 pointing out their relevance to the CLT. In Section 2.5, we incorporate a nonzero mean in the univariate version of Theorem 2, thus deriving the LDP for the empirical variance. Section 2.6 presents the LDP for the empirical autocorrelation vector of an i.i.d. process X_j as well as some counter intuitive results concerning the validity of this LDP when $\{X_j\}$ is an AR(1) process. An approach to higher order expansions is sketched in Section 2.7.

2.1. Continuous Time

Let $\{X_t\}$ be a real-valued, centered, separable stationary Gaussian process with spctral density $f(s)$, i.e., with the covariance $R(t) = E(X_0 X_t) = \int_{-\infty}^{\infty} e^{its} f(s)\,ds$.

Denote $S_T = \int_0^T X_t^2\,dt$, $M = \text{ess sup}\, f(s)$.

Theorem 1. Suppose that $\{X_t\}_{t \geqslant 0}$ has bounded spectral density function $f(s) \in L_1(\mathbb{R}, ds)$. Then $\{T^{-1}S_T\}$ satisfies the LDP with the rate function

$$I(x) = \sup_{-\infty < y < 1/(4\pi M)} \{xy - L(y)\} \tag{2.1}$$

where for $y < 1/(4\pi M)$

$$L(y) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \log(1 - 4\pi y f(s))\, ds \tag{2.2}$$

As an application, suppose that X_t is the Ornstein-Uhlenbeck process, i.e., the stationary solution to $dX_t = -aX_t + \sqrt{a}\, dW_t$, $a > 0$. The spectral density is $f(s) = (1/\pi)\, a/(a^2 + s^2)$ with $M = 1/(\pi a)$. Integrating expression in (2.2) we get $L(y) = \frac{1}{2}a - \frac{1}{2}\sqrt{a^2 - 4ay}$, leading to $I(x) = (a/4)(\sqrt{x} - 1/\sqrt{x})^2$ for $x > 0$ and $I(x) = \infty$ otherwise.

For the explicit computation of the rate function in other spectral cases, such as ARMA(p, q) processes, see [Ref. 7, Sections 2.5 and 3].

2.2. Discrete Time

The following result is the finite-dimensional discrete time version of Theorem 1.

Theorem 2. Let $\{\mathbf{X}_k\}_{k=1,2,\ldots}$ be a centered, stationary Gaussian $\mathbb{R}^d$-valued sequence with the spectral density $\mathbf{F}(s) = [F_{i,j}(s)]$ such that ess sup $\|\mathbf{F}(s)\| < \infty$ (where $\|\mathbf{F}\|$ denotes the operator norm associated with the matrix $\mathbf{F}$, c.f. (3.10). Then for every nonnegative definite symmetric real matrix $\mathbf{W}$, $\{n^{-1} \sum_{j=1}^{n} \langle \mathbf{X}_j \mid \mathbf{W}\mathbf{X}_j \rangle\}$ satisfies the LDP with rate function

$$I(x) = \sup_{-\infty < y < 1/(2M)} \{xy - L(y)\} \tag{2.3}$$

where $M = \text{ess sup}\, \|\mathbf{W}^{1/2}\mathbf{F}(s)\, \mathbf{W}^{1/2}\|$ and for $y < 1/(2M)$

$$L(y) = -\frac{1}{4\pi} \int_0^{2\pi} \log \det(I - 2y\mathbf{W}\mathbf{F}(s))\, ds \tag{2.4}$$

Remark 1. Clearly, Theorem 2 implies that the LDP holds also when $\mathbf{W}$ is a nonpositive definite symmetric real matrix. However, in Section 2.6

we give an example of **W** that is neither positive definite nor negative definite for which $L(y) = \infty$ even when all eigenvalues of $2y\mathbf{W}\mathbf{F}(s)$ are uniformly (in s) strictly less than 1.

The following special case of Theorem 2 is of interest.

Corollary 1. Let $\{X_k\}_{k=1,2,...}$ be a real-valued, centered, stationary Gaussian process with bounded spectral density function $f(s)$. Then $\{n^{-1} \sum_{j=1}^n X_j^2\}$ satisfies the LDP with the rate function of (2.3) where here $M = \text{ess sup} f(s)$ and

$$L(y) = -\frac{1}{4\pi}\int_0^{2\pi} \log(1 - 2yf(s))\, ds \tag{2.5}$$

The next corollary follows from Corollary 1 when $p = 2$ and from Ref. 15 by an approximation argument when $p < 2$ (consider Ref. 7 [Section 4.4] for details).

Corollary 2. Suppose that $\{X_k\}_{k=1,2,...}$ has continuous spectral density satisfying $\int_0^{2\pi} \log f(s)\, ds > -\infty$. If $p \leqslant 2$ then $\{n^{-1} \sum_{j=1}^n |X_j|^p\}$ satisfies the LDP.

Remark 2. Theorems 1 and 2 can also be extended to the multivariate index case (Gaussian random fields on $\mathbb{R}^k$ or $\mathbb{Z}^k$). Indeed, Ref. 18 [Chapter 8] develops the relevant tools for such an extension.

2.3. Unbounded Spectral Density

We next show that the LDP corresponding to more moderate deviations of $S_n = \sum_{j=1}^n X_j^2$ holds true even when the spectral density is unbounded.

Theorem 3. Suppose that the real-valued, centered stationary Gaussian process $\{X_j\}_{j \geqslant 1}$ has spectral density function $f(s) \in L_q(ds)$ for some $2 < q \leqslant \infty$. Let $\{m_n\}$ be such that $n^{-1/q} m_n \to \infty$ while $n^{-1/2} m_n \to 0$. Then $\{m_n(n^{-1} S_n - E(X_1^2))\}$ satisfies the LDP with speed m_n^2/n and the rate function

$$I(x) = \frac{x^2}{2\sigma^2}$$

where

$$\sigma^2 = \frac{1}{\pi} \int_0^{2\pi} f^2(s)\, ds \tag{2.6}$$

Remark 3. With minor changes in the statement and in the proof, LDPs of same speed and rate function as in Theorem 3 hold true in the multivariate setup of Theorem 2 and in the continuous time setup of Theorem 1 provided (2.1) is modified to

$$\sigma^2 = \pi^{-1} \int_0^{2\pi} tr(\mathbf{F}(s))^2\, ds \tag{2.7}$$

in the former case (taking $\mathbf{W} = \mathbf{I}$) and

$$\sigma^2 = 4\pi \int_{-\infty}^{\infty} f^2(s)\, ds \tag{2.8}$$

in the latter.

2.4. Normal Convergence

Lemmas 3 and 6 from the proofs of Theorems 1 and 2, respectively, yield the following CLTs.

Proposition 1.

(i) If $\{X_t\}$ is a real-valued, centered, separable stationary Gaussian process with the spectral density $f(s) \in L_2(\mathbb{R}, ds) \cap L_1(\mathbb{R}, ds)$, then $(1/\sqrt{T}) \int_0^T (X_t^2 - E(X_0^2))\, dt$ is asymptotically normal $N(0, \sigma)$ as $T \to \infty$ with σ^2 given by (2.8).

(ii) If $\{\mathbf{X}_k\}_{k=1,2,\ldots}$ is a centered, stationary Gaussian $\mathbb{R}^d$-valued sequence with the spectral density $\mathbf{F}(s) = [F_{i,j}(s)]$, such that $tr(\mathbf{F}(s))^2$ is integrable, then

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\langle \mathbf{X}_i \mid \mathbf{X}_i \rangle - E(\langle \mathbf{X}_1 \mid \mathbf{X}_1 \rangle))$$

is asymptotically normal $N(0, \sigma)$ as $n \to \infty$ with σ^2 given by (2.7).

For a direct proof of part (ii) in the univariate case ($d = 1$), see [Ref. 1, Thm. 2], [Ref. 17, Thm. 2]. For related non-normal convergence, see [Ref. 23]. Other related results are given in [Ref. 3, Thm. 5] and the references therein (cf also Ref. 21, Thm. 3, p. 58).

2.5. Noncentered Processes and the LDP for the Empirical Variance

By the contraction principle much of the preceding results extends to the case of noncentered stationary Gaussian processes. For example, in the context of Corollary 1 our starting point is as follows.

Proposition 2. Let $\{X_j\}$ be a real-valued centered stationary Gaussian process whose spectral density $f(\cdot)$ is differentiable. Let $\mathbf{S}_n = [\sum_{j=1}^n X_j, \sum_{j=1}^n X_j^2]'$. Then $\{n^{-1}\mathbf{S}_n\}$ satisfies the LDP (in $\mathbb{R}^2$) with the rate function

$$J(x_1, x_2) = I(x_2 - x_1^2) + \frac{x_1^2}{2f(0)} \tag{2.9}$$

where $0/0 := 0$ in (2.9) and $I(\cdot)$ is as defined by (2.3) and (2.5).

Applying the contraction principle (see [Ref. 12, Theorem 4.2.1]) with respect to the continuous function $g(x_1, x_2) = x_2 + 2x_1\mu + \mu^2$: $\mathbb{R}^2 \to \mathbb{R}$, we see that for a noncentered process $Y_j = X_j + \mu$, the sequence $\{n^{-1} \sum_{j=1}^n Y_j^2\}$ satisfies the LDP (in $\mathbb{R}$) with rate function

$$\tilde{J}(z) = \inf_{\{(x_1, x_2):\, z = g(x_1, x_2)\}} J(x_1, x_2) = \sup_{y < 1/(2M)} \left\{ zy - \frac{\mu^2 y}{1 - 2yf(0)} - L(y) \right\}$$

where $M = \text{ess sup } f(s)$ and $L(y)$ is given by (2.5) (compare also Ref. 2, [p. 361]).

Similarly, applying the contraction principle with respect to the continuous function $h(x_1, x_2) = x_2 - x_1^2$ results with the empirical variance of $\{X_j\}_{j=1}^n$ satisfying the LDP with the *same* rate function $I(\cdot)$ as for $\{n^{-1} \sum_{j=1}^n X_j^2\}$.

2.6. The Empirical Autocorrelation Vector

For $j \geq 0$, let $S_n^{(j)} = \sum_{k=1}^{n-j} X_k X_{k+j}$. Then $n^{-1} S_n^{(j)}$ is the jth empirical autocorrelation based on a sample of size n. For fixed $d \geq 1$ let $\mathbf{S}_n = [S_n^{(0)},..., S_n^{(d)}] \in \mathbb{R}^{d+1}$. If $f(\cdot)$ is the spectral density of $\{X_j\}$, denote

$$\mathbf{f}(s) = [f(s), f(s) \cos s, ..., f(s) \cos sd]' \in \mathbb{R}^{d+1}$$

Proposition 3. Suppose that $\{X_k\}_{k=1,2,...}$ are i.i.d. $N(0, 1)$ random variables. Then $\{n^{-1}\mathbf{S}_n\}$ satisfies the LDP (in $\mathbb{R}^{d+1}$) with the rate function

$$I(\mathbf{x}) = \sup\{\langle \mathbf{x} \mid \mathbf{y} \rangle - L(\mathbf{y}) : \mathbf{y} \in D\}$$

where

$$D = \{\mathbf{y} \in \mathbb{R}^{d+1}: \sup_{0 \leq s \leq 2\pi} \langle \mathbf{y} \mid \mathbf{f}(s) \rangle < 1/2\}$$

and for $\mathbf{y} \in D$

$$L(\mathbf{y}) = -\frac{1}{4\pi} \int_0^{2\pi} \log(1 - 2\langle \mathbf{y} \mid \mathbf{f}(s) \rangle)\, ds \tag{2.10}$$

Remark 4. The proof of Proposition 3 (with the same formula for the rate function) extends to any differentiable spectral density $f(s)$ provided that for all $\mathbf{y} \in D$

$$\limsup_{n \to \infty} n^{-1} \log E(\exp(\langle \mathbf{y} \mid \mathbf{S}_n \rangle)) < \infty \tag{2.11}$$

However, the following example shows that for $d = 1$ and for every AR(1) process with $0 < |a| < 1$, (2.11) is false for some $\mathbf{y} \in D$. Hence, in these cases even if $\{n^{-1}\mathbf{S}_n\}$ satisfies the LDP, the rate function cannot be given by the same expression as in Proposition 3.

Example 1. Let X_k be an AR(1) process (with $\beta_0 = 1$, $\beta_1 = 0$ and $0 < |a| < 1$) corresponding to $r_i = E[X_0 X_i] = a^i/(1 - a^2)$ for $i = 0, 1,\ldots$ and $f(s) = 1/(1 + a^2 - 2a \cos s)$. Then, $\mathbf{y} = \lambda[1 + a^2, -2a]' \in D$ for every $\lambda < 1/2$. Let $\mathbf{R}_n$ denote the covariance matrix of $\mathbf{X} = [X_1,\ldots, X_n]'$ and let $\mathbf{Y}_n$ be the $n \times n$ symmetric Toeplitz matrix corresponding to $y_0 = \lambda(1 + a^2)$, $y_1 = -\lambda a$ and $y_i = 0$ for all $1 < i \leq n - 1$. Since $\mathbf{R}_n^{-1}[r_0,\ldots, r_{n-1}]' = [1, 0,\ldots, 0]'$, we have for $\lambda > (1 - a^2)/2$ and all n large enough

$$\begin{aligned}\langle [r_0,\ldots, r_{n-1}] \mid (\mathbf{R}_n^{-1} - 2\mathbf{Y}_n)[r_0,\ldots, r_{n-1}]' \rangle \\ = r_0 - 2\lambda(1 + a^2) \sum_{i=0}^{n-1} r_i^2 + 4\lambda a \sum_{i=0}^{n-2} r_i r_{i+1} < 0\end{aligned}$$

implying that $E(\exp(\lambda(1 + a^2)\, S_n^{(0)} - 2\lambda a S_n^{(1)})) = \infty$ (see Lemma 1).

Note that this expression is related to Theorem 2. Indeed,

$$\lambda(1 + a^2)(S_n^{(0)} - \gamma X_n^2 - (1 - \gamma)\, X_1^2) - 2\lambda a S_n^{(1)} = \sum_{j=1}^{n-1} \langle \mathbf{X}_j \mid \mathbf{W}_\gamma \mathbf{X}_j \rangle$$

where $\mathbf{X}_j = [X_j, X_{j+1}]' \in \mathbb{R}^2$ and

$$\mathbf{W}_\gamma = \lambda \begin{bmatrix} \gamma(1+a^2) & -a \\ -a & (1-\gamma)(1+a^2) \end{bmatrix}$$

Considering $\lambda \geqslant 0$, $\mathbf{W}_\gamma$ is nonnegative define iff $\gamma \in [a^2/(1+a^2), 1/(1+a^2)]$. For this range of γ it follows by applying Lemma 6 to $\mathbf{Y}_j = \mathbf{W}_\gamma^{1/2} \mathbf{X}_j$ that for all $\lambda < 1/2$,

$$\lim_{n \to \infty} n^{-1} \log E(\exp(\lambda(1+a^2)(S_n^{(0)} - \gamma X_n^2 - (1-\gamma) X_1^2) - 2\lambda a S_n^{(1)}))$$
$$= -\tfrac{1}{2} \log(1-2\lambda) \tag{2.12}$$

It can also be verified that for every $\gamma > (1+a^2)$ the left side of (2.12) is infinite for some $\lambda \in (0, 1/2)$, while the eigenvalues of $\mathbf{W}_\gamma \mathbf{F}(s)$ (which are 0 and λ) are independent of γ.

Remark 5. Example 1 shows that the large deviations of the empirical autocorrelation vector are sensitive to boundary effects (the choice of γ), and that Theorem 2 does not extend to matrices $\mathbf{W}$ which are neither nonnegative definite nor nonpositive definite.

2.7. Exact Asymptotic

The following result comes essentially form Ref. 18 [p. 76]. Together with saddle point approximation, it can be used to find higher order asymptotic expansions for probabilities of "regular enough" sets in Corollary 1. We do not pursue this possibility here.

Corollary 3. Suppose $\{X_k\}_{k \geqslant 1}$ is a centered, real-valued stationary Gaussian sequence with bounded spectral density $f(s)$ and $M = \operatorname{ess\,sup} f(s)$. Let $S_n = \sum_{k=1}^n X_k^2$ and $L(y)$ be defined by (2.5). Then for all $y < 1/(2M)$ the sequence $\{\exp(-nL(y))\, E(\exp(yS_n))\}$ is monotonically nonincreasing. If in addition $f(s)$ is differentiable and for some $\alpha > 0$ the function $f'(s)$ is uniformly Lipschitz continuous with exponent α then

$$\lim_{n \to \infty} \exp(-nL(y))\, E(\exp(yS_n)) = \exp\left(L(y) - \frac{1}{2\pi} \int\int_{|z| \leqslant 1} |h_y'(z)|^2 \, d\sigma\right)$$

where

$$h_y(z) = \frac{1}{4\pi} \int_0^{2\pi} \log(1 - 2yf(s)) \frac{1 + ze^{-is}}{1 - ze^{-is}} \, ds$$

and $\sigma(dz)$ is the surface measure on the unit disc in $\mathbb{C}$.

3. PROOFS

We start with the following well known elementary result.

Lemma 1. Suppose $\mathbf{X} = [X_1, \ldots, X_n]'$ is a real valued centered Gaussian vector with the covariance matrix $\mathbf{R}$ and let $\mathbf{M}$ be a symmetric real valued $n \times n$-matrix. Then with $\lambda_1, \ldots, \lambda_n$ the eigenvalues of the matrix $\mathbf{MR}$

$$\log E \exp(z \langle \mathbf{X} \mid \mathbf{MX} \rangle) = -\tfrac{1}{2} \sum_{j=1}^{n} \log(1 - 2z\lambda_j)$$

for $z \in \mathbb{C}$ such that $\max_j \{Re(z)\, \lambda_j\} < 1/2$. Furthermore, $\log E \exp(y \langle \mathbf{X} \mid \mathbf{MX} \rangle) = \infty$ for $y \in \mathbb{R}$ such that $\max_j \{ y\lambda_j \} \geqslant 1/2$.

With $\mathbf{X} = \mathbf{R}^{1/2}\mathbf{Z}$ and $\mathbf{Z}$ a standard multivariate normal, Lemma 1 follows by direct integration of the density of $\mathbf{Z}$.

Lemma 2. If $\{Y_j\}$ are i.i.d. random variables with mean zero, finite second moment and positive probability density function at 0, then for each $\theta > 0$ there is $\delta > 0$ such that

$$\inf \left\{ P\left(\left| \sum_{i=1}^{\infty} k_i Y_i \right| < \theta \right) : \sum_{i=1}^{\infty} |k_i| \leqslant 1 \right\} \geqslant \delta$$

Proof. Denote $\sigma^2 = E(Y^2)$ and fix the sequence $\{k_i\}$. Without loss of generality, we may assume that $|k_i| \geqslant |k_{i+1}|$ for all $i \geqslant 1$. Note that then the condition $\sum_j |k_j| \leqslant 1$ implies that $|k_j| \leqslant 1/j$ for all $j \geqslant 1$. Consequently, for every $r \geqslant 1$ by Chebyshev's inequality we have

$$P\left(\left| \sum_{i=r+1}^{\infty} k_i Y_i \right| < \theta \right) \geqslant 1 - \frac{\sigma^2}{\theta^2} \sum_{j=r+1}^{\infty} \frac{1}{j^2} \tag{3.1}$$

Note that one can find $r_0 = r_0(\theta)$ such that the right-hand side of (3.1) is strictly positive. Choose now such $r_0(\theta/2)$. By independence we have

$$P\left(\left|\sum_{i=1}^{\infty} k_i Y_i\right| < \theta\right) \geqslant P\left(\left|\sum_{i=1}^{r_0} k_i Y_i\right| < \theta/2\right) P\left(\left|\sum_{i=r_0+1}^{\infty} k_i Y_i\right| < \theta/2\right)$$

and, since $|k_i| \leqslant 1$, using (3.1) we get

$$P\left(\left|\sum_{i=1}^{\infty} k_i Y_i\right| < \theta\right) \geqslant P(\max_{1 \leqslant i \leqslant r_0} |Y_i| < \theta/(2r_0))\, P\left(\left|\sum_{i=r_0+1}^{\infty} k_i Y_i\right| < \theta/2\right)$$

$$\geqslant P(|Y_1| < \theta/(2r_0))^{r_0} \left(1 - \frac{4\sigma^2}{\theta^2} \sum_{j=r_0+1}^{\infty} \frac{1}{j^2}\right) =: \delta$$

This ends the proof with $\delta > 0$ as defined. ∎

3.1. Proof of Theorem 1

Let $L_T(z) = \log E(\exp(zS_T))$ for $z \in \mathbb{C}$ with $Re(z) < 1/(4\pi M)$.
The next Lemma is motivated by a heuristic argument in Ref. 2.

Lemma 3. Under the assumptions of Theorem 1, for $Re(z) < 1/(4\pi M)$ we have

$$\lim_{T \to \infty} \frac{1}{T} L_T(z) = -\frac{1}{4\pi} \int_{-\infty}^{\infty} \log(1 - 4\pi z f(s))\, ds$$

Proof. For $T > 0$, denote by $\lambda_j = \lambda_j(T)$ the eigenvalues of

$$\int_0^T R(t-s)\, g(s)\, ds = \lambda g(t) \in L_2([0, T]) \tag{3.2}$$

and let $e_j = e_j(t) \in L_2([0, T], dt)$ be the corresponding orthonormal eigenfunctions. Since by Mercer's theorem, $R(t-s) = \sum_j \lambda_j e_j(t)\, e_j(s)$ with positive and summable eigenvalues $\{\lambda_j\}$, we have the Karhunen-Loéve expansion $X_t = \sum_j \sqrt{\lambda_j}\, \gamma_j e_j(t)$, where γ_j are i.i.d. $N(0, 1)$. Note that

$$\sup_j \lambda_j = \sup_{g \in L_2,\, \|g\| = 1} \int_0^T g(t)\, dt \int_0^T g(u)\, du \int_{-\infty}^{\infty} e^{i(t-u)s} f(s)\, ds$$

A square-integrable $g(\cdot)$ is also integrable on $[0, T]$. Thus, switching the order of integration, we get

$$\sup_j \lambda_j \leqslant M \int_{-\infty}^{\infty} \left| \int_0^T g(t)\, e^{its}\, dt \right|^2 ds = 2\pi M \tag{3.3}$$

where the last equality is by Plancherel's theorem. Therefore $Re(z) < 1/(4\pi M) \leqslant 1/(2\lambda_j)$ and

$$\begin{aligned} \frac{1}{T} L_T(z) = \frac{1}{T} \log E[\exp(zS_T)] &= -\frac{1}{2T} \sum_{j=1}^{\infty} \log(1-2z\lambda_j) \\ &= -\frac{1}{2} \int_0^{2\pi M} \log(1-2zx)\ \mu_T(dx) \end{aligned} \tag{3.4}$$

where $\mu_T(dx) := 1/T \sum_j \delta_{\lambda_j}(dx)$ denotes the distribution of the eigenvalues on $[0, 2\pi M]$. Fix z and choose $\delta > 0$ such that $2\,|z|\,\delta < 1$ and such that $\{s: 2\pi f(s) = \delta\}$ is of Lebesgue measure zero. By Ref. 18 [p. 139] for $k = 1, 2,...$ we have

$$\lim_{T \to \infty} \int_0^{2\pi M} x^k \mu_T(dx) = (2\pi)^{k-1} \int_{-\infty}^{\infty} f^k(s)\, ds \tag{3.5}$$

and also for every bounded continuous $F(\cdot)$

$$\lim_{T \to \infty} \int_\delta^{2\pi M} F(x)\, \mu_T(dx) = \frac{1}{2\pi} \int_{\{s: 2\pi f(s) \geqslant \delta\}} F(2\pi f(s))\, ds \tag{3.6}$$

Let $P_k(x)$ be the kth Taylor polynomial for $x \mapsto \log(1-2zx)$. Notice that from (3.5) and (3.6), for each fixed k we get

$$\int_0^\delta P_k(x)\, \mu_T(dx) \to \frac{1}{2\pi} \int_{\{s: 2\pi f(s) \leqslant \delta\}} P_k(2\pi f(s))\, ds \tag{3.7}$$

Clearly, for $0 \leqslant x \leqslant \delta$ we have

$$|P_k(x) - \log(1-2zx)| = \left| \sum_{j=k+1}^{\infty} (2zx)^j/j \right| < \frac{1}{k} \frac{(2x\,|z|)^{k+1}}{1-2\,|z|\,\delta} \leqslant \frac{1}{k} \frac{2x\,|z|}{1-2\,|z|\,\delta}$$

Given $\varepsilon > 0$ choose $k > 2|z| (1 - 2|z| \delta)^{-1} \varepsilon^{-1}$. Then by (3.7) choose $T_0 = T_0(k)$ such that for all $T > T_0$ we have

$$\Delta_1 := \left| \int_0^\delta P_k(x)\, \mu_T(dx) - \frac{1}{2\pi} \int_{\{s : 2\pi f(s) \leqslant \delta\}} P_k(2\pi f(s))\, ds \right| < \varepsilon$$

and by (3.5) (with $k = 1$)

$$\int_0^{2\pi M} x \mu_T(dx) < 2R(0)$$

Enlarging T_0 if necessary, by (3.6) we may also ensure that for all $T > T_0$,

$$\Delta_2 := \left| \int_\delta^{2\pi M} \log(1 - 2zx)\, \mu_T(dx) - \frac{1}{2\pi} \int_{\{s : 2\pi f(s) \geqslant \delta\}} \log(1 - 4\pi z f(s))\, ds \right| < \varepsilon$$

Therefore for all $T > T_0$ we have

$$\left| \int_0^{2\pi M} \log(1 - 2zx)\, \mu_T(dx) - \frac{1}{2\pi} \int_{-\infty}^{\infty} \log(1 - 4\pi z f(s))\, ds \right|$$
$$\leqslant \Delta_1 + \Delta_2 - \varepsilon \int_0^{2\pi M} x \mu_T(dx) + \varepsilon \int_{-\infty}^{\infty} f(s)\, ds < \varepsilon(2 + 3R(0)) \quad \blacksquare$$

Remark 6. By the induced convergence for analytic funtions, from Lemma 3 it follows that

$$T^{-1} \frac{d}{dy} L_T(y) \to \frac{d}{dy} L(y) = \int_{-\infty}^{\infty} \frac{f(s)}{1 - 4\pi y f(s)}\, ds$$

for all $y < 1/(4\pi M)$ (this can also be verified directly using Ref. 18 [p. 139]).

Remark 7. Let $\lambda_1(T)$ be the maximal eigenvalue of (3.2). Then $\lambda_1(T) \leqslant 2\pi M$ by (3.3), and therefore by Ref. 18 [p. 139] one has $\lambda_1(T) \to 2\pi M$ as $T \to \infty$.

Proof of Theorem 2.1. By Remark 7 and Lemma 1 it follows that $L(y) = \lim_{T \to \infty} T^{-1} L_T(y)$ is infinite for $y > 1/(4\pi M)$. Lemma 3 implies that $L(y)$ exists and given by (2.2) for $y < 1/(4\pi M)$. Define $L(1/(4\pi M)) := \lim_{y \nearrow 1/(4\pi M)} L(y)$ (which by monotone convergence coincides with

$L(1/(4\pi M))$ of (2.2), and note that by the monotonicity of $L_T(y)$ with respect to y

$$\liminf_{T\to\infty,\, y_T\to 1/(4\pi M)} T^{-1}L_T(y_T) \geqslant L(1/(4\pi M)) \tag{3.8}$$

The required LDP follows by the Gärtner-Ellis Theorem (see Ref. 12 [Thm. 2.3.6]) when $L(1/(4\pi M)) = \infty$. Indeed, then (3.8) holds with equality and $c := \lim_{y \nearrow 1/(4\pi M)} (d/dy)\, L(y) = \infty$ so that $L(\cdot)$ is steep. However, in general this is not the case (and it is not even clear that $T^{-1}L_T(1/(4\pi M))$ converges), so we follow instead the strategy of parameter dependent change of measure, as outlined in Ref. 13. By the monotonicity of $L_T(\cdot)$ it follows that [Ref. 13, (2.13) and (2.15)] hold true. Excluding the trivial case of zero spectral density, since $L'(y) > 0$ is non-decreasing, there is $c > 0$ such that $L'(y) \to c$ as $y \nearrow 1/(4\pi M)$. Examining [Ref. 13, Prop. 2.14] we see that the LDP with the rate function of (2.1) holds even for $L(1/(4\pi M)) < \infty$ as soon as $c = \infty$ (i.e., when $L(\cdot)$ is steep). Turning to deal with $c < \infty$, observe that then $I(\cdot)$ of (2.1) is continuous at $x = c$ and it is easy to check that for $x \geqslant c$

$$I(x) = \frac{x}{4\pi M} - L\left(\frac{1}{4\pi M}\right)$$

Thus, by [Ref. 13, Prop. 2.14], suffices to show that for all $x > c$ and all $\varepsilon > 0$ small enough

$$\liminf_{T\to\infty} \frac{1}{T} \log P(|T^{-1}S_T - x| < \varepsilon) \geqslant -\frac{1}{4\pi M}(x+\varepsilon) + L\left(\frac{1}{4\pi M}\right) \tag{3.9}$$

in order to complete the proof of the theorem. To this end, let $\lambda_1(T) \geqslant \lambda_2(T) \geqslant \cdots \geqslant \lambda_n(T) \geqslant \cdots$ be the eigenvalues of (3.2) and for $y < 1/(2\lambda_1)$ let

$$k_j(y, T) = \frac{\lambda_j}{T(1 - 2y\lambda_j)}$$

Since $T^{-1}(d/dy)\, L_T(y) = \sum_j k_j(y, T)$ is monotone in y and approaches ∞ as y approaches $1/(2\lambda_1)$, there exists $y_T < 1/(2\lambda_1(T))$ such that $\sum_{j=1}^{\infty} k_j = x$ for $k_j = k_j(y_T, T)$. Moreover, for each fixed $y < 1/(4\pi M)$, by Remark 1 $\lim_T T^{-1}(d/dy)\, L(y) = (d/dy)\, L(y) \leqslant c < x$, while $\limsup_T y_T \leqslant 1/(4\pi M)$ by Remark 2; hence $y_T \to 1/(2\pi M)$. For y_T as earlier, define the masure Q_T via

$$\frac{dQ_T}{dP} = \exp(y_T S_T - L_T(y_T))$$

and let V_T denote the random variable $(T^{-1}S_T - x)$ under measure Q_T. Note that by (3.3) the Laplace transform of V_T is given by

$$E[e^{sV_T}] = \prod_{i=1}^{\infty} \exp(-sk_i)/\sqrt{1-sk_i}$$

where $k_i = k_i(y_T, T)$. Therefore V_T has the representation

$$V_T = \sum_{j=1}^{\infty} k_j(Z_j^2 - 1)$$

with Z_j i.i.d. normal N(0, 1). By Lemma 2 we now deduce that $Q_T(|T^{-1}S_T - x| < \varepsilon) \geqslant \delta$ for all $\varepsilon > 0$ and some $\delta = \delta(\varepsilon) > 0$ which is independent of T. Since $y_T \geqslant 0$ for all large T,

$$\begin{aligned} &T^{-1} \log P(|T^{-1}S_T - x| < \varepsilon) \\ &= T^{-} \log\left(\int \frac{dP}{dQ_T} 1_{|T^{-1}S_T - x| < \varepsilon}\, dQ_T\right) \\ &\geqslant T^{-1} \log Q_T(|T^{-1}S_T - x| < \varepsilon) - y_T(x+\varepsilon) + T^{-1}L_T(y_T) \end{aligned}$$

and in the limit $T \to \infty$ the required lower bound of (3.9) follows from (3.8). ∎

3.2. Proof of Theorem 2

Throughout this proof we consider $\mathbb{R}^n$, $n \geqslant 1$ as Hilbert subspaces of ℓ_2 with the inherited norms. For an $n \times n$-matrix **A**, we consider the usual operator norm

$$\|\mathbf{A}\| = \sup_{\mathbf{y} \in \mathbb{R}^n \setminus 0} \frac{\|\mathbf{Ay}\|}{\|\mathbf{y}\|} \tag{3.10}$$

and the Hilbert-Schmidt norm $|\mathbf{A}| = \sqrt{\operatorname{tr}(\mathbf{AA}')}$ (with the usual convention that $\mathbf{A}'$ is the conjugate transpose of the matrix **A**). It is well known that $|\mathbf{ABC}| \leqslant \|\mathbf{A}\| \cdot |\mathbf{B}| \cdot \|\mathbf{C}\|$, and that $\|\mathbf{A}\| \leqslant |\mathbf{A}|$, see e.g., [Ref. 16, Section XI.6]. The distribution of the eigenvalues $\{\lambda_1, ..., \lambda_n\}$ of **A** is the discrete probability measure

$$\mu_n(dx) = n^{-1} \sum_{j=1}^{n} \delta_{\lambda_j}(dx)$$

(either on $\mathbb{R}$ or on $\mathbb{C}$, depending on whether **A** is symmetric, or not). The following result is known.

Lemma 4 ([Ref. 18, p. 105]). Suppose the $n \times n$ matrices $\mathbf{A}_n$ and $\mathbf{B}_n$ have the distribution of the eigenvalues μ_n and ν_n respectively and assume that

$$\sup_n(\|\mathbf{A}_n\| + \|\mathbf{B}_n\|) < \infty \tag{3.11}$$

and

$$\lim_{n\to\infty} n^{-1} |\mathbf{A}_n - \mathbf{B}_n|^2 = 0 \tag{3.12}$$

Then $\lim_{n\to\infty} |\int x^k \mu_n(dx) - \int x^k \nu_n(dx)| = 0$ for every $k = 1, 2,...$

Let $\mathbf{R}_n = \operatorname{cov}(\mathbf{X}_0, \mathbf{X}_n)$ be the $d \times d$-covariance matrices, and let μ_n be the distribution of the eigenvalues of the block-Toeplitz $nd \times nd$ matrix

$$\mathbf{A}_n = \begin{bmatrix} \mathbf{R}_0 & \mathbf{R}_1 & \dots & \mathbf{R}_{n-1} \\ \mathbf{R}_{-1} & \mathbf{R}_0 & \dots & \mathbf{R}_{n-2} \\ \vdots & & \ddots & \vdots \\ \mathbf{R}_{-(n-1)} & \mathbf{R}_{-(n-2)} & \dots & \mathbf{R}_0 \end{bmatrix} \tag{3.13}$$

Extending the argument of Ref. 18 [p. 113] we next provide the asymptotic of μ_n.

Lemma 5. If $M = \operatorname{ess\,sup} \|\mathbf{F}(s)\| < \infty$ then $\sup_n \|\mathbf{A}_n\| \leqslant M$. Moreover, for any $a < b$ such that $m(s\colon \lambda_j(s) = a) = m(s\colon \lambda_j(s) = b) = 0$ for $j = 1,..., d$,

$$\lim_{n\to\infty} \mu_n([a, b]) = (2\pi d)^{-1} \sum_{j=1}^{d} m(s\colon a < \lambda_j(s) < b) \tag{3.14}$$

where m is Lebesgue measure on $[0, 2\pi]$ and $\lambda_1(s) \geqslant \lambda_2(s) \geqslant \dots \lambda_d(s) \geqslant 0$ are the eigenvalues of $\mathbf{F}(s)$ (recall that $\mathbf{F}(s)$, $0 \leqslant s \leqslant 2\pi$, are Hermitian, non-negative definite matrices).

Proof. For $(n-1)/2 \geqslant A \geqslant 1$ ler $\hat{\mathbf{R}}_k = (1 - k/A)\,\mathbf{R}_k$ for $k = 0,..., A$ and $\hat{\mathbf{R}}_k = 0$ for $k > A$, with $\hat{\mathbf{R}}_{-k} = \hat{\mathbf{R}}'_k$. Let $\mathbf{B}_{n,A}$ be the block-Toeplitz $nd \times nd$ matrix constructed as in (3.13) but with the blocks $\hat{\mathbf{R}}_k$ instead of $\mathbf{R}_k$. Let $\mathbf{C}_{n,A}$ be the block-circulant matrix associated with $\mathbf{B}_{n,A}$ by using the blocks $\hat{\mathbf{R}}_{k \bmod n}$ instead of $\mathbf{R}_k$. Let $\mathbf{F}_A(s) = \sum_{k=-A}^{A} e^{-iks} \hat{\mathbf{R}}_k$, with $\{\lambda_{j,k}\}_{j=1,...,d}$ denoting the eigenvalues of $\mathbf{F}_A(2\pi k/n)$, $k = 0,..., n-1$ and $\mathbf{v}_{j,k} \in \mathbb{R}^d$ the corresponding eigenvectors. The usual argument for circulant matrices shows that for $j=1,\dots,d$, $k=0,\dots,n-1$ the nd-dimensional vectors

$$(\mathbf{v}_{j,k},\, e^{2\pi ik/n}\mathbf{v}_{j,k},\dots,\, e^{2\pi ik(n-1)/n}\mathbf{v}_{j,k})$$

are linearly independent eigenvectors of $\mathbf{C}_{n,A}$ corresponding to the eigenvalues $\lambda_{j,k}$; therefore those are all the eigenvalues of $\mathbf{C}_{n,A}$. Consequently, $\|\mathbf{C}_{n,A}\| \leqslant \sup_s \|\mathbf{F}_A(s)\|$ and since

$$\mathbf{F}_A(s)=\frac{1}{2\pi}\int_0^{2\pi}\frac{\sin^2(A(s-t)/2)}{A\sin^2((s-t)/2)}\,\mathbf{F}(t)\,dt$$

clearly,

$$\sup_s \|\mathbf{F}_A(s)\| \leqslant \sup_s \|\mathbf{F}(s)\|\cdot\frac{1}{2\pi}\int_0^{2\pi}\frac{\sin^2(At/2)}{A\sin^2(t/2)}\,dt=M \tag{3.15}$$

We turn now to prove that $\|\mathbf{A}_n\| \leqslant M$ and $\|\mathbf{B}_{n,A}\| \leqslant M$. To this end, fix n, pick $\mathbf{x}_j\in\mathbb{R}^d$ and write $\mathbf{x}=(\mathbf{x}_j)$ as a column vector. Then,

$$\begin{aligned}\langle \mathbf{x}\,|\,\mathbf{A}_n\mathbf{x}\rangle &= (2\pi)^{-1}\int_0^{2\pi}\left\langle \sum_{k=1}^n e^{-iks}\mathbf{x}_k \,\Big|\, \mathbf{F}(s)\sum_{m=1}^n e^{ims}\mathbf{x}_m\right\rangle ds\\ &\leqslant (2\pi)^{-1}\int_0^{2\pi}\left\|\sum_{k=1}^n e^{-iks}\mathbf{x}_k\right\|^2 \|\mathbf{F}(s)\|\,ds\\ &\leqslant \sup_{0\leqslant s\leqslant 2\pi}\|\mathbf{F}(s)\|\left(\frac{1}{2\pi}\int_0^{2\pi}\left\|\sum_{k=1}^n e^{-iks}\mathbf{x}_k\right\|^2 ds\right)=M\,\|\mathbf{x}\|^2\end{aligned}$$

By a similar argument we have for $n>A$

$$\begin{aligned}\langle \mathbf{x}\,|\,\mathbf{B}_{n,A}\mathbf{x}\rangle &= (2\pi)^{-1}\int_0^{2\pi}\left\langle \sum_{k=1}^n e^{-iks}\mathbf{x}_k \,\Big|\, \mathbf{F}_A(s)\sum_{m=1}^n e^{ims}\mathbf{x}_m\right\rangle ds\\ &\leqslant \|\mathbf{x}\|^2 \sup_s \|\mathbf{F}_A(s)\| \leqslant M\,\|\mathbf{x}\|^2\end{aligned}$$

This shows that the matrices $\mathbf{A}_n$, $\mathbf{B}_{n,A}$ and $\mathbf{C}_{n,A}$ satisfy (3.11) for every choice of $A\leqslant (n-1)/2$.

Applying Parseval's relation elementwise one has

$$\sum_{j=-\infty}^{\infty}|\mathbf{R}_j|^2=(2\pi)^{-1}\int_0^{2\pi}|\mathbf{F}(s)|^2\,ds\leqslant dM^2$$

Since for every $n > A$ we have

$$n^{-1} |\mathbf{A}_n - \mathbf{B}_{n,A}|^2 \leqslant 2 \sum_{j=1}^{A} (j/A)^2 |\mathbf{R}_j|^2 + 2 \sum_{j=A+1}^{\infty} |\mathbf{R}_j|^2$$

by Kronecker's Lemma it follows that $n^{-1} |\mathbf{A}_n - \mathbf{B}_{n,A}|^2$ can be made arbitrarily small (uniformly in $n > A$) by choosing A large enough. Therefore, by choosing first A large and then n large enough, we can make sure that (3.12) holds both for $|\mathbf{A}_n - \mathbf{B}_{n,A}|$ and for $|\mathbf{B}_{n,A} - \mathbf{C}_{n,A}|$ since

$$|\mathbf{B}_{n,A} - \mathbf{C}_{n,A}|^2 \leqslant 2A \sum_{j=1}^{A} |\mathbf{R}_j|^2 \leqslant AdM^2$$

Consequently, by Lemma 4 the asymptotic of μ_n is the same as the asymptotic of the distribution of the eigenvalues of $\mathbf{C}_{n,A}$ provided we let $n \to \infty$ first and then take $A \to \infty$.

Fix a positive integer ℓ. In view of the continuity of $\mathbf{F}_A(s)$ we have for any fixed $A \geqslant 1$

$$\lim_{n \to \infty} n^{-1} \sum_{k=1}^{n} \operatorname{tr}(\mathbf{F}_A(2\pi k/n)^\ell) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{tr}(\mathbf{F}_A(s)^\ell)\, ds$$

Also

$$\begin{aligned} &\left| (2\pi)^{-1} \int_0^{2\pi} \operatorname{tr}(\mathbf{F}_A(s)^\ell - \mathbf{F}(s)^\ell)\, ds \right|^2 \\ &\quad \leqslant d(2\pi)^{-1} \int_0^{2\pi} |\mathbf{F}_A(s)^\ell - \mathbf{F}(s)^\ell|^2\, ds \\ &\quad \leqslant d\ell^2 M^{2(\ell-1)} (2\pi)^{-1} \int_0^{2\pi} |\mathbf{F}_A(s) - \mathbf{F}(s)|^2\, ds \end{aligned}$$

and since,

$$(2\pi)^{-1} \int_0^{2\pi} |\mathbf{F}_A(s) - \mathbf{F}(s)|^2\, ds = 2 \sum_{j=1}^{A} (j/A)^2 |\mathbf{R}_j|^2 + 2 \sum_{j=A+1}^{\infty} |\mathbf{R}_j|^2$$

we have for $A \to \infty$ that $\int_0^{2\pi} \operatorname{tr}(\mathbf{F}_A(s)^\ell - \mathbf{F}(s)^\ell)\, ds \to 0$, leading to

$$\lim_{A \to \infty} \lim_{n \to \infty} n^{-1} \sum_{k=1}^{n} \operatorname{tr}(\mathbf{F}_A(2\pi k/n)^\ell) = \frac{1}{2\pi} \int_0^{2\pi} \operatorname{tr}(\mathbf{F}(s)^\ell)\, ds$$

With this holding for every positive integer ℓ, the limit (3.14) follows by Ref. 18 [p. 105]. ▮

Let $S_n = \sum_{j=1}^{n} \langle \mathbf{X}_j \mid \mathbf{X}_j \rangle$ and $L_n(z) = \log E(\exp(zS_n))$ for $z \in \mathbb{C}$.

Lemma 6. If $\sup_s \|\mathbf{F}(s)\| = M < \infty$, then for every $z \in \mathbf{C}$ such that $Re\, z < 1/(2M)$,

$$\lim_{n \to \infty} n^{-1} L_n(z) = -\frac{1}{4\pi} \int_0^{2\pi} \log \det(I - 2z\mathbf{F}(s))\, ds \tag{3.16}$$

Remark 8. For $d = 1$ this lemma is known, see Ref. 9 [p. 105], or Ref. 10 [Example 3.1a)].

Proof. Clearly,

$$S_n = [\mathbf{X}_1,..., \mathbf{X}_n][\mathbf{X}_1,..., \mathbf{X}_n]'$$

Therefore by Lemma 1, for $Re(z) < 1/(2 \max_j \lambda_j)$

$$n^{-1} L_n(z) = -1/(2n) \sum_{j=1}^{nd} \log(1 - 2z\lambda_j)$$

where $\{\lambda_j\}$ are the eigenvalues of the symmetric nonnegative definite matrix $\mathbf{A}_n$.

Lemma 5 implies that $\max_j \lambda_j = \|\mathbf{A}_n\| \leqslant M$ for all n, and by (3.14) actually $\|\mathbf{A}_n\| \to M$ as $n \to \infty$. Consequently, (3.16) follows by applying (3.14) and observing that

$$n^{-1} L_n(z) = -\frac{d}{2} \int_0^M \log(1 - 2zx)\, \mu_n(dx) \quad ▮$$

Remark 9. By the induced convergence for analytic functions, from Lemma 6 it follows that for $y < 1/(2M)$

$$n^{-1} \frac{d}{dy} L_n(y) \to \frac{d}{dy} L(y) = \frac{1}{2\pi} \sum_{j=1}^{d} \int_0^{2\pi} \frac{\lambda_j(s)}{1 - 2y\lambda_j(s)}\, ds$$

where $\lambda_j(s), j = 1,..., d$ are the (nonnegative) eigenvalues of $\mathbf{F}(s)$. (This claim can also be verified directly from (3.14).

Proof of Theorem 2. For $\mathbf{W}$ an identity matrix, the proof repeats the reasoning from the proof of Theorem 1. Indeed, by Lemma 6, $n^{-1}L_n(y)$ onverges to $L(y)$ of (2.4) for $y < 1/(2M)$, while by Lemmas 1 and 5, for $y > 1/(2M)$

$$L(y) = \lim_{n \to \infty} n^{-1}L_n(y) = \infty$$

Excluding the trivial case of zero spectral density, notice that $L'(y) > 0$ is monotonically increasing for $y < 1/(2M)$, and let $c > 0$ be such that $L'(y) \to c$ as $y \nearrow 1/(2M)$. Define $L(1/(2M)) = \lim_{y \nearrow 1/(2M)} L(y)$. Since [Ref. 13, (2.13) and (2.15)] hold by the monotonicity of $L_n(\cdot)$, if $c = \infty$, then $L(y)$ is steep and the LDP with the rate function $I(\cdot)$ of (2.3) and (2.4) follows by [Ref. 13, Prop. 2.14] (even if $n^{-1}L_n(1/(2M))$ fails to converge). Otherwise, $c < \infty$ and $I(x)$ is continuous at $x = c$ with $I(x) = x/(2M) - L(1/(2M))$ for all $x \geqslant c$. Letting $\{\lambda_j\}$ denote the nonnegative eigenvalues of the matrix $\mathbf{A}_n$, the n-dependent change of measure via $dQ_n/dP = \exp(y_n S_n - L_n(y_n))$ results with $n^{-1}S_n - x$ (under Q_n) having the representation $\sum_{j=1}^{nd} k_j(Z_j^2 - 1)$ with Z_j i.i.d. normal $N(0, 1)$ and $k_j = \lambda_j/(n(1 - 2y_n\lambda_j))$, where $y_n < 1/(2 \max_j \lambda_j)$ chosen such that $\sum_{j=1}^{nd} k_j = x$. Since $\max_j\{\lambda_j\} = \|\mathbf{A}_n\| \to M$ as $n \to \infty$ it follows by Remark 9, that $\lim_n y_n = 1/(2M)$ and the proof of the large deviations lower bound for $x \geqslant c$ is completed by applying Lemma 2 (note that $\liminf_n n^{-1}L_n(y_n) \geqslant L(1/(2M))$). For any $\mathbf{W}$ nonnegative definite symmetric real matrix, we have $\mathbf{W} = \mathbf{W}^{1/2}\mathbf{W}^{1/2}$ with $\mathbf{W}^{1/2}$ also nonnegative symmetric real matrix. Hence $\langle \mathbf{X}_j \mid \mathbf{W}\mathbf{X}_j \rangle = \langle \mathbf{Y}_j \mid \mathbf{Y}_j \rangle$ for $j = 1, 2,...,$ where $\mathbf{Y}_j = \mathbf{W}^{1/2}\mathbf{X}_j$ is a stationary process of bounded spectral density $\mathbf{W}_{1/2}\mathbf{F}(s)\,\mathbf{W}^{1/2}$. Therefore, the general case follows by applying the above proof of the process $\{\mathbf{Y}_j\}$. ∎

Remark 10. For $d = 1$, by Lemma 1 and Ref. 18 [pp. 38, 44], $n^{-1} \log E(\exp((2M)^{-1} \sum_{j=1}^{n} X_j^2))$ converges as $n \to \infty$ to $L(1/(2M))$ of (2.5). The validity of this result in the general context of Theorem 2 is not addressed here.

3.3. Proof of Theorem 3

We first bound the maximal eigenvalue of $\mathbf{A}_n$—the covariance matrix of $[X_1,..., X_n]'$.

Lemma 7. If $f \in L_q(ds)$ for $1 \leqslant q \leqslant \infty$ then $\|\mathbf{A}_n\| \leqslant Cn^{1/q}$ for some $C < \infty$ and all $n > 1$.

Proof. Let $\mathbf{x} = [x_1, ..., x_n]'$ be such $\|\mathbf{x}\| = 1$ and $\|\mathbf{A}_n\| = \langle \mathbf{x} \mid \mathbf{A}_n \mathbf{x} \rangle$. Then, denoting $1/p + 1/q = 1$, we have $\|\mathbf{A}_n\| = 1/2\pi \int_0^{2\pi} f(s) \, |\sum x_j e^{ijs}|^2 \, ds \leqslant \|f\|_q (1/2\pi \int_0^{2\pi} |\sum x_j e^{ijs}|^{2p} \, ds)^{1/p} \leqslant C(\sum |x_j|)^{(2p-2)/p} \leqslant Cn^{1/q}$. ▮

Proof of Theorem 3. Let $T_n = m_n(n^{-1}S_n - EX_1^2)$ and $\lambda_j = \lambda_j(n)$, $1 \leqslant j \leqslant n$, denote the eigenvalues of $\mathbf{A}_n$. Since by Lemma 7 and the choice of m_n $\max_j \lambda_j/m_n \to 0$, for every $y \in \mathbb{R}$ and all $n \geqslant n_0(y)$ we have

$$\log E \exp(nm_n^{-2} yT_n) = -ynm_n^{-1} EX_1^2 - \tfrac{1}{2} \sum_{j=1}^{n} \log(1 - 2y\lambda_j/m_n)$$

Notie that by Taylor's Theorem for $|w| < 1$

$$\log(1 - w) = -w - (1/2)\, w^2(1 - tw)^{-2}$$

where $t = t(w) \in [0, 1]$. This is applied here to $w_j = 2y\lambda_j/m_n$ which by Lemma 7 satisfies $\sup_j |w_j| \to 0$ as $n \to \infty$, and hence, $|1 - t(w_j)\, w_j| \to 1$ uniformly in $1 \leqslant j \leqslant n$. Thus, the limit of

$$m_n^2 n^{-1} \log E \exp(nm_n^{-2} yT_n)$$

is the same as that of

$$ym_n\left(n^{-1} \sum_{j=1}^{n} \lambda_j - E(X_1^2)\right) + y^2 n^{-1} \sum_{j=1}^{n} \lambda_j^2$$

Clearly, $\sum_{j=1}^{n} \lambda_j = \operatorname{tr} \mathbf{A}_n = nE(X_1^2)$, and

$$n^{-1} \sum_{j=1}^{n} \lambda_j^2 = n^{-1} \operatorname{tr} \mathbf{A}_n^2 = \sum_{k=-(n-1)}^{n-1} (1 - |k|/n)\, r_k^2 = \sum_{k=-(n-1)}^{n-1} r_k^2 - 2 \sum_{k=1}^{n-1} (k/n)\, r_k^2$$

Notice that by Parseval's identity $\sum_{k=-(n-1)}^{n-1} r_k^2 \to \sum_{k=-\infty}^{\prime} r_k^2 = \sigma^2/2$ as $n \to \infty$. On the other hand, by Kronecker's Lemma $\sum_{k=1}^{n-1} (k/n)\, r_k^2 \to 0$ as $n \to \infty$ leading to

$$\lim_{n \to \infty} m_n^2 n^{-1} \log E \exp(nm_n^{-2} yT_n) = \tfrac{1}{2} y^2 \sigma^2$$

We complete the proof by applying the Gärtner-Ellis Theorem for the speed $a_n = m_n^2/n \to 0$ (cf. [Ref. 12, Thm. 2.3.6 and Remark (a)]. ▮

3.4. Proof of Proposition 1

For $f(s)$ or $\|\mathbf{F}(s)\|$ bounded, the CLT follows immediately from Lemmas 3 and 6 by a simple complex analysis argument given in [Ref. 4, Prop. 1]. In general, for every $M<\infty$, we let $X_t = Y_t + Z_t$ in the continuous time setup and $\mathbf{X}_k = \mathbf{Y}_k + \mathbf{Z}_k$ in the discrete time setup; in the former case Y_t and Z_t are independent, real-valued, centered, separable stationary Gaussian processes with spectral densities $f_y(s) = \min(f(s), M)$ and $f_z(s) = f(s) - f_y(s)$, while in the latter $\mathbf{Y}_k$ and $\mathbf{Z}_k$ are independent, $\mathbf{R}^d$-valued, centered, stationary Gaussian sequences, with the spectral densities $\mathbf{F}_y(s)$ and $\mathbf{F}_z(s)$ having the same eigenvectors as $\mathbf{F}(s)$ but with eigenvalues $\min(\lambda_j(s), M)$ and $\max(\lambda_j(s) - M, 0)$ respectively. Then, in the continuous time setup,

$$\begin{aligned} W_M &:= \frac{1}{\sqrt{T}} \int_0^T (X_t^2 - Y_t^2 - E(X_0^2 - Y_0^2))\, dt \\ &= \frac{1}{\sqrt{T}} \int_0^T (Z_t^2 - E(Z_0^2))\, dt + \frac{2}{\sqrt{T}} \int_0^T Y_t Z_t\, dt \end{aligned}$$

has mean zero and variance bounded above by $\varepsilon_M := 4\sigma(4\pi \int_{-\infty}^{\infty} f_z(s)^2\, ds)^{1/2}$, while in the discrete time setup,

$$W_M := \frac{1}{\sqrt{n}} \sum_{i=1}^{n} (\langle X_i \mid \mathbf{X}_i \rangle - \langle \mathbf{Y}_i \mid \mathbf{Y}_i \rangle - E(\langle \mathbf{X}_0 \mid \mathbf{X}_0 \rangle - \langle \mathbf{Y}_0 \mid \mathbf{Y}_0 \rangle))$$

has zero mean and variance bounded by $\varepsilon_M := 4\sigma(\pi^{-1} \int_0^{2\pi} \operatorname{tr}(\mathbf{F}_z(s)^2)\, ds)^{1/2}$. Note that in both cases $\varepsilon_M \to 0$ as $M \to \infty$, hence for every $\delta > 0$, by Chebyshev's inequality $P(|W_M| > \delta) < \varepsilon_M/\delta^2 \to 0$ as $M \to \infty$ uniformly in $T(n)$. Since $f_y(s)$ is bounded, $1/\sqrt{T} \int_0^T (Y_t^2 - E(Y_0^2))\, dt$ is asymptotically normal $N(0, \sigma_M)$ as $T \to \infty$, with $\sigma_M := (4\pi \int_{-\infty}^{\infty} f_y(s)^2\, ds)^{1/2}$ monotonically increasing to σ as $M \to \infty$. Similarly, in the discrete time setup, $\|\mathbf{F}_y(s)\|$ is bounded and hence $1/\sqrt{n} \sum_{i=1}^n (\langle \mathbf{Y}_i \mid \mathbf{Y}_i \rangle - E(\langle \mathbf{Y}_0 \mid \mathbf{Y}_0 \rangle))$ is asymptotically normal $N(0, \sigma_M)$ as $n \to \infty$, with $\sigma_M := (\pi^{-1} \int_0^{2\pi} \operatorname{tr}(\mathbf{F}_y(s))^2\, ds)^{1/2} \nearrow \sigma$ as $M \to \infty$. The required CLT then follows by the continuity of the normal distribution function. ∎

3.5. Proof of Proposition 2

For $\mathbf{y} = [y_1, y_2]$ define $L_n(\mathbf{y}) = \log E \exp(\langle \mathbf{y} \mid \mathbf{S}_n \rangle)$. Let $\mathbf{R}_n$ be the covariance matrix of $\mathbf{X} = [X_1, ..., X_n]'$ with $\lambda_1(n)$ denoting the maximal eigenvalue of $\mathbf{R}_n$, $\mathbf{I}_n$ denoting the identity matrix, and $\mathbf{e}_n = [1, 1, ..., 1]'$. By adapting the calculations of Lemma 1 we have for $y_2 < 1/(2\lambda_1(n))$

$$L_n(\mathbf{y}) = L_n([0, y_2]) + \tfrac{1}{2} y_1^2 \langle \mathbf{e}_n \mid \mathbf{R}_n^{1/2}(\mathbf{I}_n - 2y_2\mathbf{R}_n)^{-1} \mathbf{R}_n^{1/2}\mathbf{e}_n \rangle$$

(and $L_n(\mathbf{y}) = \infty$ for all other values of $\mathbf{y}$).

Lemma 8. If $y_2 < 1/(2M)$ then

$$L(\mathbf{y}) = \lim_{n \to \infty} n^{-1} L_n(\mathbf{y}) = L(y_2) + \frac{y_1^2 f(0)}{2(1 - 2y_2 f(0))}$$

with $L(y)$ given by (2.5), and $n^{-1}L_n(\mathbf{y}) \to \infty$ when $y_2 > 1/(2M)$.

Proof. We have by Ref. 18 [p. 65] that $n^{-1}L_n([0, y_2]) \to L(y_2)$ for all $y_2 < 1/(2M)$ and $n^{-1}L_n(\mathbf{y}) \to \infty$ for all $y_2 > 1/(2M)$. Taking $y_2 < 1/(2M)$ we have by Ref. 18 [pp. 27, 53, 209] that

$$n^{-1}\langle \mathbf{e}_n \mid (\mathbf{I}_n - 2y_2\mathbf{R}_n)^{-1}\mathbf{e}_n \rangle \to 1/(1 - 2y_2 f(0))$$

and the proof is completed by noting that $2y_2\mathbf{R}_n^{1/2}(\mathbf{I}_n - 2y_2\mathbf{R}_n)^{-1}\mathbf{R}_n^{1/2} + \mathbf{I}_n = (\mathbf{I}_n - 2y_2\mathbf{R}_n)^{-1}$. ∎

Proof of Proposition 2. Defining $L([y_1, 1/(2M)]) = \lim_{y_2 \nearrow 1/(2M)} L(\mathbf{y})$ and $L(\mathbf{y}) = \infty$ for $y_2 > 1/(2M)$, it is easy to check that $J(x_1, x_2)$ of (2.9) is the Fenchel-Legendre transform of $L(\mathbf{y})$. Here again, it is easy to check that conditions [Ref. 13, (2.13) and (2.15)] follow from the monotonicity of $L_n(\mathbf{y})$ with respect to y_2. Hence, suffices to show that $L(\mathbf{y})$ is steep, for then the LDP with rate function $J(\cdot)$ holds by [Ref. 13, Prop. 2.14] (even if $n^{-1}L_n(\mathbf{y})$ fails to converge for $y_2 = 1/(2M)$). To that end, note that for $y_2 < 1/(2M)$

$$\frac{\partial L(\mathbf{y})}{\partial y_2} \geqslant \frac{1}{2\pi} \int_0^{2\pi} \frac{f(s)}{1 - 2y_2 f(s)}\, ds$$

Hence, by the differentiability of $f(s)$ we have $\partial L(\mathbf{y})/\partial y_2 \to \infty$ as $y_2 \nearrow 1/(2M)$ implying that $L(\mathbf{y})$ is steep (for more details, see the proof of Proposition 3). ∎

3.6. Proof of Proposition 3

Let $L_n(\mathbf{y}) = \log E(\exp(\langle \mathbf{y} \mid \mathbf{S}_n \rangle))$ and let $\mathbf{Y}_n$ be the symmetric Toeplitz $n \times n$-matrix whose first row is $(y_0, \frac{1}{2}y_1, ..., \frac{1}{2}y_d, 0, ..., 0)$. Let $\mathbf{R}_n$ be the covariance matrix of $\mathbf{X} = [X_1, ..., X_n]'$. Since $\langle \mathbf{y} \mid \mathbf{S}_n \rangle = \mathbf{X}'\mathbf{Y}_n\mathbf{X}$, by Lemma 1 we have

$$L_n(\mathbf{y}) = -1/2 \sum_{j=1}^{n} \log(1 - 2\lambda_j(\mathbf{y})) \tag{3.17}$$

where $\lambda_j(\mathbf{y})$ are the eigenvalues of the matrix $\mathbf{M}_n = \mathbf{Y}_n \mathbf{R}_n$ and $\mathbf{y}$ is such that $\max_j\{\lambda_j(\mathbf{y})\} < 1/2$.

For i.i.d. X_j of unit variance we have that $\mathbf{R}_n$ is the identity matrix, hence $\mathbf{M}_n = \mathbf{Y}_n$ is the symmetric Toeplitz matrix corresponding to the "signed" bounded spectral density $\langle \mathbf{y} \mid \mathbf{f}(s) \rangle$. In particular, by Ref. 18 [p. 65] for $\mathbf{y} \in D$

$$\frac{1}{n} L_n(\mathbf{y}) \to L(\mathbf{y}) = -\frac{1}{4\pi} \int_0^{2\pi} \log(1 - 2\langle \mathbf{y} \mid \mathbf{f}(s) \rangle)\, ds$$

By Ref. 18 [pp. 38, 44] this relation holds for $\mathbf{y} \in \partial D$, i.e., when $\sup_s \langle \mathbf{y} \mid \mathbf{f}(s) \rangle = \frac{1}{2}$, while $n^{-1} L_n(\mathbf{y}) \to \infty$ for all other values of $\mathbf{y}$.

Notice that if $\|\mathbf{y}\| < 1/(2(d+1))$ then $\mathbf{y} \in D$. Therefore, in order to establish the LDP, we need only to verify the steepness condition, i.e.,

$$\lim_{\mathbf{y} \to \mathbf{y}_0,\, \mathbf{y} \in D} \|L'(\mathbf{y})\| = \infty$$

for all $\mathbf{y}_0 \in \partial D$, see Ref. 12 [Thm. 2.3.6]. To this end, fix $\mathbf{y}_0 \in \partial D$ and let $0 \leqslant s_0 \leqslant 2\pi$ be such that $\langle \mathbf{y}_0 \mid \mathbf{f}(s_0) \rangle = 1/2$. It suffices to show that $|\langle \mathbf{y}_0 \mid L'(\mathbf{y}) \rangle| \to \infty$ as $\mathbf{y} \to \mathbf{y}_0$, $\mathbf{y} \in D$. Clearly,

$$\langle \mathbf{y}_0 \mid L'(\mathbf{y}) \rangle = \frac{1}{2\pi} \int_0^{2\pi} \frac{\langle \mathbf{y}_0 \mid \mathbf{f}(s) \rangle}{1 - 2\langle \mathbf{y} \mid \mathbf{f}(s) \rangle}\, ds$$

Let $I_+ = \{s: \langle \mathbf{y}_0 \mid \mathbf{f}(s) \rangle \geqslant 0\}$, and $I_- = \{s: \langle \mathbf{y}_0 \mid \mathbf{f}(s) \rangle < 0\}$. We have

$$\limsup_{\mathbf{y} \to \mathbf{y}_0} \left| \frac{1}{2\pi} \int_{I_-} \frac{\langle \mathbf{y}_0 \mid \mathbf{f}(s) \rangle}{1 - 2\langle \mathbf{y} \mid \mathbf{f}(s) \rangle}\, ds \right| \leqslant \|\mathbf{y}_0\|$$

Since $\mathbf{f}(s)$ is differentiable, for each $\varepsilon > 0$ there is $\delta > 0$ such that for $|s - s_0| < \delta$ we have $|\langle \mathbf{y}_0 \mid \mathbf{f}(s) \rangle - \langle \mathbf{y}_0 \mid \mathbf{f}(s_0) \rangle| < \varepsilon\delta$ and $\langle \mathbf{y}_0 \mid \mathbf{f}(s) \rangle \geqslant m > 0$; i.e. $(s_0 - \delta, s_0) \subset I_+$ (if $s_0 = 0$ replace $(s_0 - \delta, s_0)$ by $(s_0, s_0 + \delta)$). Then

$$\int_{I_+} \frac{\langle \mathbf{y}_0 \mid \mathbf{f}(s)\rangle}{1-2\langle \mathbf{y} \mid \mathbf{f}(s)\rangle}\, ds \geqslant \int_{s_0-\delta}^{s_0} \frac{\langle \mathbf{y}_0 \mid \mathbf{f}(s)\rangle}{1-2\langle \mathbf{y} \mid \mathbf{f}(s)\rangle}\, ds$$

$$\geqslant m \int_{s_0-\delta}^{s_0} \frac{1}{2\langle \mathbf{y}_0 \mid \mathbf{f}(s_0)-\mathbf{f}(s)\rangle + 2\langle \mathbf{y}_0-\mathbf{y} \mid \mathbf{f}(s)\rangle}\, ds$$

$$\geqslant m \frac{\delta}{2\varepsilon\delta + 2\,\|\mathbf{y}_0-\mathbf{y}\|}$$

Therefore $\liminf_{\mathbf{y}\to\mathbf{y}_0}\langle \mathbf{y}_0 \mid L'(\mathbf{y})\rangle \geqslant m/(4\pi\varepsilon) - \|\mathbf{y}_0\|$. Taking $\varepsilon \to 0$, this ends the proof. ∎

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