ide 1

Conditional moments, gamma, free gamma, and free Poisson laws

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Abstract

This talk is based on joint paper with M. Bożejko *"On a class of free Lévy laws related to a regression problem"*.

ide 2

Notation and Plan of talk

$\boxed{\boxed{n=1}}$	X, Y random variables	$S = X+Y$
$\downarrow$		
$\boxed{1<n<\infty}$	$\mathbf{X}, \mathbf{Y}$ random symmetric $n \times n$ matrices	$\mathbf{S} = \mathbf{X}+\mathbf{Y}$
$\downarrow$		
$\boxed{\boxed{n=\infty}}$	$\mathbb{X}, \mathbb{Y}$ LLN limits of $\mathbf{X}/n, \mathbf{Y}/n$, aka noncommutative random variables	$\mathbb{S} = \mathbb{X}+\mathbb{Y}$

Slide 3

1 Univariate gamma law

Suppose $X, Y > 0$ are **non-degenerate** independent random variables. Let $S = X+Y$ be their sum and $Z = \frac{X}{S}$ be the quotient.

Theorem 1 ([Lukacs, 1955]) *If S and Z are independent, then X is gamma with density $\frac{1}{\Gamma(p)} x^{p-1} e^{-x}$, $x > 0$, $p > 0$ after normalization.*

$\boxed{\text{Simple proof}}$ Use conditional moments. If Z and S are independent then $E(X|S) = E(ZS|S) = SE(Z)$ and $E(X^2|S) = E(Z^2S^2|S) = S^2E(Z^2)$

Thus $E(X|S) = \mu S$ and $E(X^2|S) = (\sigma^2+\mu^2)S^2$. Or

$$E(X|S) = \mu S \text{ and } \operatorname{Var}(X|S) = \sigma^2 S^2$$

Slide 4

$\boxed{\text{Can we determine the distribution of } X \text{ from } E(X|S) \text{ and } \operatorname{Var}(X|S)\text{?}}$

Yes, as noticed in [Wesołowski, 1989].

Theorem 2 *[Laha and Lukacs, 1960] Suppose X, Y are independent, $E(X) = E(Y) = 0$, $E(X^2) = E(Y^2) = 1$, $S = X+Y$.*

$$E(X|S) = \frac{1}{2}S,$$

and for some constants C, a, b

$$\operatorname{Var}(X|S) = C\left(1 + \frac{a}{2}S + \frac{b}{4}S^2\right).$$

Then X and Y have the classical Meixner type law. In particular, $C = 1/(2+b)$ and X is ...

ide 5

$E(X|S) = \frac{1}{2}S$, $\mathrm{Var}(X|S) = C(1 + \frac{a}{2}S + \frac{b}{4}S^2)$.

(i) Normal (Gaussian), if $a = b = 0$;

(ii) $\boxed{\textit{Poisson type, if } b = 0 \textit{ and } a \neq 0;\ \lambda = 1/a^2}$

(iii) Pascal (negative binomial) type, if $b > 0$ and $a^2 > 4b$;

(iv) $\boxed{\boxed{\textit{Gamma type, if } b > 0 \textit{ and } a^2 = 4b;\ p = 1/b}}$

(v) Meixner type, if $b > 0$ and $a^2 < 4b$;

(vi) Binomial type, if $b = -1/n$ and n is an integer.

Gamma-type $\frac{1}{2+b}(1 + \frac{a}{2}S + \frac{b}{4}S^2) = \frac{a^2}{16+2a^2}(\frac{4}{a} + S)^2$

Note: Conversely, centered standardized Meixner laws have these conditional moments.

Note: [Morris, 1982] exponential families, orthogonal polynomials

ide 6

2 Matrix gamma: Wishart law

Suppose $\mathbf{X}, \mathbf{Y}$ are **non-degenerate** independent symmetric semi-positive-definite matrices. Consider

$$\mathbf{S} = \mathbf{X} + \mathbf{Y},\ \mathbf{Z} = \mathbf{S}^{-1/2}\mathbf{X}\mathbf{S}^{-1/2}$$

[Olkin and Rubin, 1962], [Casalis and Letac, 1996], [Bobecka and Wesołowski, 2002] prove

Theorem 3 *If $\mathbf{S} > 0$, $\mathbf{X}, \mathbf{Y}$ are not concentrated on the same one-dimensional subspace, the law of $\mathbf{Z}$ is invariant under orthogonal transformations, or $\mathbf{X}$, $\mathbf{Y}$ have strictly positive twice-differentiable densities, and $\mathbf{Z}$ and $\mathbf{S}$ are independent, then $\mathbf{X}$ is Wishart, $E(\exp\langle\theta, \mathbf{X}\rangle) = \det(\mathbf{I} - \theta)^{-p}$, $p > \frac{n-1}{2}$, after re-normalization.*

Slide 7

Simplest proof? For any i.i.d. matrices,

$$E(\mathbf{X}|\mathbf{S}) = \frac{1}{2}\mathbf{S}$$

If $\mathbf{X}, \mathbf{Y}$ are independent Wishart matrices with shape parameters p, q, then [Letac and Massam, 1998] show that there are $a = a(p,q)$, $b = b(p,q)$ such that

$$\mathrm{Var}(\mathbf{X}|\mathbf{S}) = a(\mathrm{tr}\mathbf{S})\mathbf{S} + b\mathbf{S}^2$$

The "simplest proof" fails. A "simple proof" in [Letac and Massam, 1998] relies on the quadratic regression property of **other** quadratic functions of $\mathbf{X}$.

Slide 8

2.1 Large Wishart matrices

$\mathbf{X}_n, \mathbf{Y}_n$ are i.i.d. $n \times n$ Wishart matrices with shape parameter $p > (n-1)/2$. $\mathbf{S}_n = \mathbf{X}_n + \mathbf{Y}_n$.

Goal: What can we say about the limit as $n \to \infty$, $p \to \infty$, $p/n \to \lambda/2 > 0$?

$E(\frac{\mathbf{X}_n}{n}|\frac{\mathbf{S}_n}{n}) = \frac{1}{2}\frac{\mathbf{S}_n}{n}$ in the limit gives $\mathcal{E}(\mathbb{X}|\mathbb{S}) = \frac{1}{2}\mathbb{S}$

$$E\left(\left(\frac{\mathbf{X}_n}{n}\right)^2 \middle| \frac{\mathbf{S}_n}{n}\right) = \frac{np}{16p^2+4p-2}\frac{\mathbf{S}_n}{n}\mathrm{tr}_n\frac{\mathbf{S}_n}{n} + \frac{4p^2+2p-1}{16p^2+4p-2}\left(\frac{\mathbf{S}_n}{n}\right)^2$$

$E(\exp\langle\theta, \mathbf{S}_n\rangle) = \det(I-\theta)^{-2p}$, so $E\exp\alpha\mathrm{tr}_n\frac{\mathbf{S}_n}{n} = (1-\alpha/n^2)^{-2pn}$. So $\mathrm{tr}_n\left(\frac{\mathbf{S}_n}{n}\right) \to \lambda$ in prob.

$$\mathrm{Var}(\mathbb{X}|\mathbb{S}) = \frac{1}{8}\mathbb{S},\ \text{or } \mathrm{Var}(2\mathbb{X}|2\mathbb{S}) = \frac{1}{4}(2\mathbb{S})$$

Slide 5: Conditional variance of $2\mathbb{X}$ is like Poisson, not like gamma!

ide 9

3 What are non-commutative r.v.?

Self-adjoint elements $\mathbb{X} = \mathbb{X}^*$ of a complex *-algebra $\mathcal{A}$ with identity; preferably, von Neumann algebra.

- State $\mathcal{E} : \mathcal{A} \to \mathbb{C}$. Faithful. Tracial; preferably, normal normalized positive linear functional: $\mathcal{E}(a^*) = \overline{\mathcal{E}(a)}$, $\mathcal{E}(\mathbb{I}) = 1$, $\mathcal{E}(aa^*) \geq 0$. $\mathcal{E}(aa^*) = 0$ implies $a = 0$. $\mathcal{E}(ab) = \mathcal{E}(ba)$; preferably, continuous in weak*-topology.
- Law of $\mathbb{X}$: probability measure μ such that $\mathcal{E}(\mathbb{X}^n) = \int_{\mathbb{R}} x^n \mu(dx)$

Example $\mathcal{A}$=random $n \times n$ matrices with $\mathcal{E}(a) = E(\mathrm{tr}_n(a))$. $\mathbb{X}, \mathbb{Y}, \ldots$ are random Hermitian matrices Voiculescu's theorem [Dykema, 1993] says that as $n \to \infty$ Hermitian matrices $\mathbb{X}/n$, $\mathbb{Y}/n$ with i.i.d entries are asymptotically free.
[Capitaine and Casalis, 2004] show that independent Wishart matrices are asymptotically free. **Note:** $\mathbb{X}\mathbb{Y}$ is not a r.v.!

le 10

3.1 Free random variables

$\mathbb{Z}, \mathbb{S}$ are free if

$$\mathcal{E}\left(\underbrace{p_1(\mathbb{S})}\, q_1(\mathbb{Z}) p_2(\mathbb{S}) q_2(\mathbb{Z}) \ldots p_{k-1}(\mathbb{S}) q_{k-1}(\mathbb{Z})) p_k(\mathbb{S}) \underbrace{q_k(\mathbb{Z})}\right) = 0$$

for all polynomials $p_1, q_1, \ldots, p_k, q_k$ such that $\mathcal{E}(p_j(\mathbb{S})) = 0$, $\mathcal{E}(q_j(\mathbb{Z})) = 0$. For example, if $\mathbb{Z}, \mathbb{S}$ are centered and free, then

$$\mathcal{E}(\mathbb{Z}\mathbb{S}\mathbb{Z}\mathbb{S}) = 0.$$

So commuting free random variables are boring! If centered:

$$0 = \mathcal{E}(\mathbb{Z}^2\mathbb{S}^2) = \mathcal{E}((\mathbb{Z}^2 - m_2\mathbb{I})\mathbb{S}^2) + m_2\mathcal{E}(\mathbb{S}^2) = m_2(\mathbb{Z})m_2(\mathbb{S})$$

If not centered: $\mathrm{Var}(\mathbb{Z})\mathrm{Var}(\mathbb{S}) = 0$.

See [Voiculescu, 2000]. Combinatorial approach [Speicher, 1997].

Slide 11

3.2 Conditional expectations

Let $\mathcal{B} \subset \mathcal{A}$ be a $*$-subalgebra. The conditional expectation is a linear map $\mathcal{E}_{\mathcal{B}} : \mathcal{A} \to \mathcal{B}$ such that $\mathcal{E}_{\mathcal{B}}(\mathbb{Y}_1\mathbb{X}\mathbb{Y}_2) = \mathbb{Y}_1\mathcal{E}_{\mathcal{B}}(\mathbb{X})\mathbb{Y}_2$ for all $\mathbb{X} \in \mathcal{A}$, $\mathbb{Y}_1, \mathbb{Y}_2 \in \mathcal{B}$. **Note:** here $\mathbb{X}, \mathbb{Y}$ are not r.v.!

Properties: Probabilistic notation: $\mathcal{E}_{\mathcal{B}}(\mathbb{X}) = \mathcal{E}(\mathbb{X}|\mathcal{B})$.

(i) If $\mathbb{X} \in \mathcal{A}, \mathbb{Y} \in \mathcal{B} \subset \mathcal{A}$, then $\mathcal{E}(\mathbb{X}\mathbb{Y}) = \mathcal{E}(\mathcal{E}(\mathbb{X}|\mathcal{B})\mathbb{Y})$

(ii) If random variables $\mathbb{U}, \mathbb{V} \in \mathcal{A}$ are free then $\mathcal{E}(\mathbb{U}|\mathbb{V}) = \mathcal{E}(\mathbb{U})\mathbb{I}$.

(iii) Let $\mathbb{W}$ be a (self-adjoint) element of the von Neumann algebra $\mathcal{B}_{\mathbb{V}}$ generated by a self-adjoint $\mathbb{V} \in \mathcal{A}$. If for all $n \geq 1$ we have $\mathcal{E}(\mathbb{U}\mathbb{V}^n) = \mathcal{E}(\mathbb{W}\mathbb{V}^n)$ then $\mathcal{E}(\mathbb{U}|\mathbb{V}) = \mathbb{W}$.

(iv) If $\mathcal{E}(\mathbb{U}_1\mathbb{V}^n) = \mathcal{E}(\mathbb{U}_2\mathbb{V}^n)$ for all $n \geq 1$, then $\mathcal{E}(\mathbb{U}_1|\mathbb{V}) = \mathcal{E}(\mathbb{U}_2|\mathbb{V})$.

Slide 12

Proof

(i) See [Takesaki, 1972]. (iv) Apply (iii). (ii) If $\mathbb{Z}$ is in the von Neumann algebra generated by $\mathbb{V}$, then $\mathcal{E}((\mathbb{U} - c\mathbb{I})\mathbb{Z}) = \mathcal{E}(\mathbb{U} - c\mathbb{I})\mathcal{E}(\mathbb{Z})$. Applying this to $\mathbb{Z} = \mathcal{E}(\mathbb{U}|\mathbb{V}) - \mathcal{E}(\mathbb{U})\mathbb{I}$ and $c = \mathcal{E}(\mathbb{U})$ after taking into account (i) we get $\mathcal{E}(\mathbb{Z}^2) = \mathcal{E}\left(\mathbb{Z}(\mathcal{E}(\mathbb{U}|\mathbb{V}) - c\mathbb{I})\right) = \mathcal{E}\left(\mathbb{Z}\mathcal{E}(\mathbb{U} - c\mathbb{I}|\mathbb{V})\right) = \mathcal{E}(\mathbb{Z}(\mathbb{U} - c\mathbb{I})) = \mathcal{E}(\mathbb{Z})\mathcal{E}(\mathbb{U} - c\mathbb{I}) = 0$. Thus $\mathcal{E}(\mathbb{U}|\mathbb{V}) = \mathcal{E}(\mathbb{U})\mathbb{I}$.

(iii) Let $\mathbb{W}' = \mathcal{E}(\mathbb{U}|\mathbb{V})$. Then $\mathcal{E}\left((\mathbb{W} - \mathbb{W}')p(\mathbb{V})\right) = 0$ for all polynomials p. Since polynomials $p(\mathbb{V})$ are dense in the von Neumann algebra generated by $\mathbb{V}$, and $\mathcal{E}(\cdot)$ is normal, this implies that $\mathcal{E}((\mathbb{W} - \mathbb{W}')(\mathbb{W} - \mathbb{W}')^*) = 0$; by faithfulness of $\mathcal{E}(\cdot)$ we deduce that $\mathbb{W}' = \mathbb{W}$.

le 13

4 Free version of Olkin-Rubin Theorem

Proposition 4 ([Bożejko and Bryc, 2004]) *Suppose random variables $\mathbb{X}, \mathbb{Y} \in \mathcal{A}$ are free, identically distributed, $\sigma = \sqrt{\mathrm{Var}(\mathbb{X})} > 0$, and such that $\mathbb{S} = \mathbb{X} + \mathbb{Y}$ is strictly positive; in particular, $m = \mathcal{E}(\mathbb{X}) > 0$. Let $\mathbb{Z} = \mathbb{S}^{-1/2}\mathbb{X}\mathbb{S}^{-1/2}$. If $\mathbb{Z}$ and $\mathbb{S}$ are free, then $\mathbb{X}$ has free-Poisson type law $\mu_{a,0}$ with $a = \sigma/m$.*

For converse, see [Capitaine and Casalis, 2004, Corollary 7.2].

Simple proof

By exchangeability, $\mathcal{E}(\mathbb{X}|\mathbb{S}) = \mathbb{S}/2$. Also $\mathcal{E}(\mathbb{Z}) = 1/2$, as $\mathcal{E}(\mathbb{Z}) = \mathcal{E}(\mathbb{S}^{-1/2}\mathbb{Y}\mathbb{S}^{-1/2})$ and $\mathcal{E}(\mathbb{S}^{-1/2}(\mathbb{X}+\mathbb{Y})\mathbb{S}^{-1/2}) = \mathcal{E}(\mathbb{I}) = 1$.

We now verify that $\mathrm{Var}(\mathbb{X}|\mathbb{S})$ is a linear function of $\mathbb{S}$. Denote the centering operation by $\mathbb{U}^\circ = \mathbb{U} - \mathcal{E}(\mathbb{U})\mathbb{I}$.

le 14

Using tracial property and freeness of $\mathbb{S}, \mathbb{Z}$:

$$\underline{\mathcal{E}(\mathbb{X}^2\mathbb{S}^m)} = \mathcal{E}(\mathbb{Z}\mathbb{S}\mathbb{Z}\mathbb{S}^{m+1}) = \mathcal{E}(\mathbb{Z}\mathbb{S}(\mathbb{Z}^\circ + \mathbf{1}/2\mathbb{I})\mathbb{S}^{m+1})$$

$$= \frac{1}{2}\mathcal{E}(\mathbb{Z}\mathbb{S}^{m+2}) + \mathcal{E}(\mathbb{Z}(\mathbb{S}^\circ + 2m\mathbb{I})\mathbb{Z}^\circ\mathbb{S}^{m+1})$$

$$= \frac{1}{4}\mathcal{E}(\mathbb{S}^{m+2}) + 2m\mathcal{E}(\mathbb{Z}\mathbb{Z}^\circ\mathbb{S}^{m+1}) + \mathcal{E}(\mathbb{Z}\mathbb{S}^\circ\mathbb{Z}^\circ\mathbb{S}^{m+1})$$

$$= \underline{\frac{1}{4}\mathcal{E}(\mathbb{S}^{m+2}) + 2m\mathrm{Var}(\mathbb{Z})\mathcal{E}(\mathbb{S}^{m+1})} + \mathcal{E}(\mathbb{Z}\mathbb{S}^\circ\mathbb{Z}^\circ\mathbb{S}^{m+1}).$$

The last term vanishes by freeness:

$$\mathcal{E}(\mathbb{Z}\mathbb{S}^\circ\mathbb{Z}^\circ\mathbb{S}^{m+1}) = \mathcal{E}((\mathbb{Z}^\circ + 1/2\mathbb{I})\mathbb{S}^\circ\mathbb{Z}^\circ\mathbb{S}^{m+1})$$

$$= 1/2\mathcal{E}(\mathbb{Z}^\circ)\mathcal{E}(\mathbb{S}^{m+2}) + \mathcal{E}(\mathbb{Z}^\circ\mathbb{S}^\circ\mathbb{Z}^\circ\mathbb{S}^{m+1})$$

$$= 0 + \mathcal{E}(\mathbb{Z}^\circ\mathbb{S}^\circ\mathbb{Z}^\circ)\mathcal{E}(\mathbb{S}^{m+1}) = 0.$$

Slide 15

Therefore,

$$\mathcal{E}(\mathbb{X}^2\mathbb{S}^m) = \mathcal{E}\left(\left(\frac{1}{4}\mathbb{S}^2 + 2m\mathrm{Var}(\mathbb{Z})\mathbb{S}\right)\mathbb{S}^m\right),$$

which by Slide 11 (iii) implies that

$$\mathcal{E}(\mathbb{X}^2|\mathbb{S}) = \frac{1}{4}\mathbb{S}^2 + 2m\mathrm{Var}(\mathbb{Z})(\mathbb{S} - 2m) + 4m^2\mathrm{Var}(\mathbb{Z})\mathbb{I}.$$

Passing to standardized random variables $\mathbb{X}^\circ/\sigma, \mathbb{Y}^\circ/\sigma$, we get

$$\mathrm{Var}(\frac{1}{\sigma}\mathbb{X}^\circ|\mathbb{S}) = \frac{m^2\mathrm{Var}(\mathbb{Z})}{\sigma^2}\left(4\mathbb{I} + 2\frac{\sigma}{m}\mathbb{S}^\circ/\sigma\right).$$

This shows that standardized random variables satisfy

$$\mathrm{Var}(\mathbb{X}_s|\mathbb{S}) = C(\mathbb{I} + \frac{a}{2}\mathbb{S}_s)$$

with $a = \sigma/m$. (This also determines $\mathrm{Var}(\mathbb{Z}) = \sigma^2/(8m^2)$.)

Slide 16

Can we determine the distribution of $\mathbb{X}$ from $\mathcal{E}(\mathbb{X}|\mathbb{S})$ and $\mathrm{Var}(\mathbb{X}|\mathbb{S})$?

Theorem 5 ([Bożejko and Bryc, 2004]) *Suppose $\mathbb{X}, \mathbb{Y} \in \mathcal{A}$ are free, self-adjoint, $\mathcal{E}(\mathbb{X}) = \mathcal{E}(\mathbb{Y}) = 0$, $\mathcal{E}(\mathbb{X}^2) = \mathcal{E}(\mathbb{Y}^2) = 1$, $\mathbb{S} = \mathbb{X} + \mathbb{Y}$ and*

$$\mathcal{E}(\mathbb{X}|\mathbb{S}) = \frac{1}{2}\mathbb{S} \tag{1}$$

and there are numbers $C, a, b \in \mathbb{R}$ such that

$$\mathrm{Var}(\mathbb{X}|\mathbb{S}) = C(\mathbb{I} + \frac{a}{2}\mathbb{S} + \frac{b}{4}\mathbb{S}^2). \tag{2}$$

Then $\mathbb{X}$ and $\mathbb{Y}$ have the free Meixner law. In particular, $b \geq -1$, $C = 1/(2+b)$ and the law of $\mathbb{X}$ is ...

le 17

$\mathrm{Var}((\mathbb{X}|\mathbb{S}) = C(\mathbb{I} + \frac{a}{2}\mathbb{S} + \frac{b}{4}\mathbb{S}^2)$.

(i) *the Wigner's semicircle law if* $a = b = 0$;

(ii) *the free Poisson type law if* $b = 0$ *and* $a \neq 0$; $\lambda = 1/a^2$

(iii) *the free Pascal type law if* $b > 0$ *and* $a^2 > 4b$;

(iv) *the free gamma type law if* $b > 0$ *and* $a^2 = 4b$;

(v) *the pure free Meixner type law if* $b > 0$ *and* $a^2 < 4b$;

(vi) *the free binomial type law if* $-1 \leq b < 0$.

Note: (i)-(v) are $\boxplus$-infinitely divisible. But this is **not** the Berkovici-Pata correspondence!

le 18

Standardized free Meixner laws

$$\int_{\mathbb{R}} \frac{1}{z-y}\mu_{a,b}(dy) = \frac{(1+2b)z + a - \sqrt{(z-a)^2 - 4(1+b)}}{2(bz^2 + az + 1)}, \qquad (3)$$

The absolutely continuous part of $\mu_{a,b}$ is

$$\frac{\sqrt{4(1+b) - (x-a)^2}}{2\pi(bx^2 + ax + 1)}$$

on $a - 2\sqrt{1+b} \leq x \leq a + 2\sqrt{1+b}$; the measure may also have an atom if $a^2 > 4b \geq 0$, and a second atom if $-1 \leq b < 0$. See [Saitoh and Yoshida, 2001]

Slide 19

5 Free Olkin-Rubin Theorem II

Proposition 6 ([Bożejko and Bryc, 2004]) *Suppose* $\mathbb{X}, \mathbb{Y} \in \mathcal{A}$ *are free, identically distributed,* $\sigma = \sqrt{\mathrm{Var}(\mathbb{X})} > 0$ *and such that* $\mathbb{S} = \mathbb{X} + \mathbb{Y}$ *is strictly positive; in particular,* $m = \mathcal{E}(\mathbb{X}) > 0$. *Let* $\mathbb{Z} = \mathbb{S}^{-1}\mathbb{X}^2\mathbb{S}^{-1}$. *If* $\mathbb{Z}$ *and* $\mathbb{S}$ *are free, then* $\mathbb{X}$ *has free-gamma type law* μ_{2a,a^2} *with* $a = \sigma/m$.

Simple proof: By exchangeability, $\mathcal{E}(\mathbb{X}|\mathbb{S}) = \mathbb{S}/2$. By freeness, $\mathcal{E}(\mathbb{X}^2|\mathbb{S}) = \mathbb{S}\mathcal{E}(\mathbb{Z}|\mathbb{S})\mathbb{S} = \mathcal{E}(\mathbb{Z})\mathbb{S}^2$. Thus

$$\mathrm{Var}(\mathbb{X}|\mathbb{S}) = c\mathbb{S}^2,$$

where $c = \mathcal{E}(\mathbb{Z}) - 1/4 \geq 0$. After standardization Slide 16 (2) holds with $a = 2\sigma/m$, $b = \sigma^2/m^2$. (And $c = \sigma^2/(2m^2 + \sigma^2)$.)

Slide 20

6 Questions and Speculations

(i) Why $n = 1$ and $n = \infty$ are simpler?

(ii) If $\mathbb{X}, \mathbb{Y}$ are free gamma, $\mathbb{S} = \mathbb{X} + \mathbb{Y}$, are $\mathbb{S}$ and $\mathbb{S}^{-1}\mathbb{X}^2\mathbb{S}^{-1}$ indeed free?

(iii) Does the matrix version of Olkin-Rubin II hold? Are there nontrivial i.i.d. $n \times n$ symmetric random matrices $\mathbf{X}, \mathbf{Y}$ with independent sum $\mathbf{S} = \mathbf{X} + \mathbf{Y}$ and quotient $\mathbf{Z} = \mathbf{S}^{-1}\mathbf{X}^2\mathbf{S}^{-1}$?

(iv) Is there a matrix version of [Laha and Lukacs, 1960]?

(v) [Laha and Lukacs, 1960] holds in more generality. Does the free version generalize, too?

le 21

Question: Are there i.i.d. symmetric random matrices $\mathbf{X}, \mathbf{Y}$ with independent $\mathbf{S} = \mathbf{X} + \mathbf{Y}$ and $\mathbf{Z} = \mathbf{S}^{-1}\mathbf{X}^2\mathbf{S}^{-1}$?

p%-trivial answers:

100% Let $\mathbf{X} = \mathrm{diag}(X_1, .., X_n)$ where X_j are independent gamma

95% A distribution of $\mathbf{X}$ is a 95%-trivial answer to the query, if one can produce from it a **new** answer by taking $\mathbf{X}' = U\mathbf{X}U^*$ for a fixed deterministic orthogonal matrix U.

Example: The Wishart distribution on the Lorenz cone i.e.
$\mathbf{X} = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$ defined on $\Omega = \mathbb{R}^2$ with the density
$f(x, y) = Ce^{-ax-by}(x^2 - y^2)^{p-1}$, where $x > |y|$, $p > 0$, $a, b > 0$ is a 95%-trivial answer.

0% Invariant under orthogonal transformations.

le 22

Theorem 7 ([Laha and Lukacs, 1960]) *Suppose X, Y are independent, $E(X) = E(Y) = 0$, $E(X^2) > 0, E(Y^2) > 0$, $S = X + Y$, and for some constants C, a, b, ρ*

$$E(X|S) = \rho S,\ \mathrm{Var}(X|S) = C(1 + \frac{a}{2}S + \frac{b}{4}S^2).$$

Then X and Y have the classical Meixner type law. In particular, X is as in Slide 4.

Question: Does the free version hold in more generality, too? If $\mathbb{X}, \mathbb{Y}$ are free, non-degenerate, $\mathcal{E}(\mathbb{X}|\mathbb{S}) = \rho\mathbb{S}$, $\mathrm{Var}(\mathbb{X}|\mathbb{S}) = C(1 + a\mathbb{S} + b\mathbb{S}^2)$, does the analogous six-part conclusion from Slide 17 follow?

Slide 23

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Slide 24

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