Free Exponential Families

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Canonical Parametrization

$$\kappa(\theta) = \ln \int_{\mathbb{R}} \exp(\theta x)\nu(dx).$$

Definition

The natural exponential family generated by ν is

$$\mathcal{F}(\nu) := \left\{ P_\theta(dx) = e^{\theta x - \kappa(\theta)}\nu(dx) : \theta \in (C, D) \right\}.$$

Re-parametrization

- $\kappa(\theta) = \ln \int_{\mathbb{R}} \exp(\theta x)\nu(dx)$ is strictly convex
- $\kappa' : (C, D) \to (A, B)$ is invertible

$$\kappa'(\psi(m)) = m \text{ and } \psi(\kappa'(\theta)) = \theta$$

Here $m \in (A, B)$, $\theta \in (C, D)$.

Definition

$$\mathcal{F}(\nu) = \left\{ W(m, dx) := P_{\psi(m)}(dx),\ m \in (A, B) \right\} \tag{1}$$

Parametrization by the mean

$m = \kappa'(\theta) = \int_{\mathbb{R}} x P_\theta(dx) \in (A, B)$. So $\int_{\mathbb{R}} x W(m, dx) = m$.

Definition

The variance function $V : (A, B) \to \mathbb{R}$ is

$$V(m) = \int (x - m)^2 W(m, dx) = \kappa''(\psi(m)).$$

Theorem (Mora)

The variance function V together with (A, B) determines $\mathcal{F}(\nu)$ uniquely.

Notation: $\mathcal{F}(V)$

Normal family

- Generating measure $\nu = e^{-x^2/2}/\sqrt{2\pi}$
- $\kappa(\theta) = \theta^2/2$ so

$$\mathcal{F}(\nu) = \left\{e^{\theta x - x^2/2 - \theta^2/2} dx/\sqrt{2\pi} : \theta \in \mathbb{R}\right\} = \left\{e^{-(x-\theta)^2/2} dx/\sqrt{2\pi} : \theta$$

- Parametrization by the mean:

$$\mathcal{F}(\nu) = \left\{e^{-(x-m)^2/2} dx/\sqrt{2\pi} : \theta \in \mathbb{R}\right\}$$

- Variance function $V(m) = 1$

Theorem

If an exponential family $\mathcal{F}$ has $V(m) = 1$ for all real m, then $\mathcal{F}$ is as above.

Poisson family

- generating measure $\nu = \sum_{k=0}^{\infty} \frac{1}{k!}\delta_k$
- $\kappa(\theta) = e^{\theta}$ so

$$\mathcal{F}(\nu) = \left\{ \sum_{k=0}^{\infty} e^{\theta k - e^{\theta}} \frac{1}{k!}\delta_k : \ \theta \in \mathbb{R} \right\}$$

- Parametrization by the mean: $m = e^{\theta}$, so inverse $\theta = \ln m$

$$\mathcal{F}(\nu) = \left\{ \sum_{k=0}^{\infty} e^{-m} \frac{m^k}{k!}\delta_k : \ m > 0 \right\}$$

- Variance function $V(m) = m$

Theorem

If an exponential family $\mathcal{F}$ has $V(m) = m$ for all positive m, then $\mathcal{F}$ is as above.

Theorem ([Morris, 1982],[Ismail and May, 1978])

Suppose $b \geq -1$. The natural exponential family with the variance function

$$V(m) = 1 + am + bm^2$$

consists of the following probability measures:

1. *the normal (Gaussian) law if $a = b = 0$;*
2. *the Poisson type law if $b = 0$ and $a \neq 0$;*
3. *the Pascal (negative binomial) type law if $b > 0$ and $a^2 > 4b$;*
4. *the Gamma type law if $b > 0$ and $a^2 = 4b$;*
5. *the hyperbolic type law if $b > 0$ and $a^2 < 4b$;*
6. *the binomial type law if $-1 \leq b < 0$ and $1/b \in \mathbb{Z}$.*

Free Version

Dispersion Models

For natural $\lambda = 1, 2, \dots$ let

$$\nu_\lambda(U) := (\nu * \nu * \cdots * \nu)(\lambda U)$$

ν_λ be the law of the average of λ independent random variables with law ν.

Proposition

The exponential family generated by ν_λ has variance function

$$V_\lambda(m) = \kappa_\lambda''(\psi_\lambda(m)) = \frac{V(m)}{\lambda}. \tag{2}$$

If $\frac{V(m)}{\lambda}$ is a variance function for all $0 < \lambda \leq 1$, $m \in (A, B)$, then the exponential family generated by ν consists of infinitely divisible probability laws.

Free Version

Differential equation for the density

Proposition

If ν generates the natural exponential family with the variance function $V(m)$ defined for $m \in (A, B)$, then the natural exponential family $W(m, dx) = w(m, x)\nu_\lambda(dx)$ satisfies

$$\frac{\partial w(m,x)}{\partial m} = \frac{x-m}{V(m)} w_\lambda(m,x) \tag{3}$$

▸ Proof

The finite difference analog of

$$\frac{\partial w}{\partial m} = \frac{x-m}{V(m)} w$$

is

$$\Delta_m w(m,x) = \frac{x-m}{V(m)} w(m,x),$$

where

$$(\Delta_m f)(m) := \frac{f(m) - f(m_0)}{m - m_0}.$$

The solution with initial condition $w(m_0, x) = 1$ is

$$w_{m_0}(m,x) = \frac{V(m)}{V(m) + (m - m_0)(m - x)}. \tag{4}$$

Definition ([Bryc and Ismail, 2005])

A free exponential family centered at m_0 is

$$\mathcal{F}_{m_0}(V) := \left\{ \frac{V(m)}{V(m) + (m - m_0)(m - x)} \nu(dx) : \; m \in (A, B) \right\}, \tag{5}$$

where $\nu = \nu_{m_0}$ is a compactly supported probability measure with mean $m_0 \in (A, B)$.

Variance function

Proposition ([Bryc and Ismail, 2005])

Family

$$\mathcal{F}_{m_0}(V) := \left\{ \frac{V(m)}{V(m) + (m - m_0)(m - x)} \nu(dx) : m \in (A, B) \right\},$$

is parameterized by the mean, and V is the variance function.

Skip Proof

Proof.

With $(\Delta_m f)(m) := \frac{f(m)-f(m_0)}{m-m_0}$ we have

$$\Delta_m w_{m_0}(m,x) = \frac{x-m}{V(m)} w_{m_0}(m,x). \tag{6}$$

Since $\Delta_m 1 = 0$, applying operator Δ_m to

$$\int w_{m_0}(m,x)\nu(dx) = 1 \tag{7}$$

and using (6) we get

$$\int x w_{m_0}(m,x)\nu(dx) = m.$$

$\square$

Proof Cont.

Similarly, since $\Delta_m m = 1$, applying Δ_m to

$$\int x w_{m_0}(m,x)\nu(dx) = m.$$

and using again the difference equation

$$\Delta_m w_{m_0}(m,x) = \frac{x-m}{V(m)} w_{m_0}(m,x).$$

we get

$$\int (x-m)^2 w_{m_0}(m,x)\nu(dx) = V(m). \tag{8}$$

$\square$

Back to Proposition

Uniqueness

Proposition ([Bryc and Ismail, 2005])

If V is analytic in a neighborhood of m_0 then the generating measure ν of the free exponential family $\mathcal{F}_{m_0}(V)$ is determined uniquely.

Skip Proof

Proof of Proposition 12.

For m close enough to m_0 so that $V(m) > 0$, re-write the definition (7) as

$$\int \frac{1}{\frac{V(m)}{m-m_0} + m - x}\nu(dx) = \frac{m-m_0}{V(m)}.$$

Thus with

$$z = m + \frac{V(m)}{m-m_0}, \tag{9}$$

the Cauchy-Stieltjes transform of ν is

$$G_\nu(z) = \frac{m-m_0}{V(m)}. \tag{10}$$

This determines $G_\nu(z)$ uniquely as an analytic function outside of the support of ν. □

Semicircle family

Example (Semi-circle free exponential family)

Function $V(m) \equiv 1/\lambda$ is the variance function of the free exponential family generated by the semicircle law of variance $1/\lambda$

$$\mathcal{F}_\lambda = \left\{ \pi_{m,\lambda}(dx) = \frac{\sqrt{4-\lambda x^2}}{2\pi\lambda(1+\lambda m(m-x))} 1_{x^2 \leq 2/\lambda} : m^2 < 1/\lambda \right\}. \tag{11}$$

From [Hiai and Petz, 2000, (3.2.2)] for $m \neq 0$, $\pi_{m,\lambda} = \mathcal{L}(m - mX + 1/(\lambda m))$ is the law of the affine transformation of a free Poisson random variable X with parameter $1/(\lambda m^2)$.
Since $\int \pi_{m,\lambda}(dx) = 1$ when $m^2 \leq 1/\lambda$, in contrast to classical exponential families, the interval $(A, B) \subset (-1/\sqrt{\lambda}, 1/\sqrt{\lambda})$ in (11) cannot be chosen independently of λ.

Theorem ([Bryc and Ismail, 2005])

Suppose $b \geq -1$, $m_0 = 0$. The free exponential family with the variance function

$$V(m) = 1 + am + bm^2$$

is generated by free Meixner laws

$$\nu(dx) = \frac{\sqrt{4(1+b) - (x-a)^2}}{2\pi(bx^2 + ax + 1)} 1_{(a-2\sqrt{1+b}, a+2\sqrt{1+b})} dx + p_1\delta_{x_1} + p_2\delta_{x_2}. \quad (12)$$

Skip Discrete part

$$\nu(dx) = \frac{\sqrt{4(1+b)-(x-a)^2}}{2\pi(bx^2+ax+1)} 1_{(a-2\sqrt{1+b},a+2\sqrt{1+b})} dx + p_1\delta_{x_1} + p_2\delta_{x_2}.$$

The discrete part of ν is absent except:

- if $b=0, a^2>1$, then $p_1 = 1-1/a^2$, $x_1=-1/a$, $p_2=0$.
- if $b>0$ and $a^2>4b$, then $p_1 = \max\left\{0, 1-\frac{|a|-\sqrt{a^2-4b}}{2b\sqrt{a^2-4b}}\right\}$, $p_2=0$, and $x_1 = \pm\frac{|a|-\sqrt{a^2-4b}}{2b}$ with the sign opposite to the sign of a.
- if $-1\leq b<0$ then there are two atoms at

$$x_{1,2} = \frac{-a\pm\sqrt{a^2-4b}}{2b},\ p_{1,2} = 1+\frac{\sqrt{a^2-4b}\mp a}{2b\sqrt{a^2-4b}}.$$

If $V(m) = 1 + am + bm^2$ then up to the type ν is

1. the Wigner's semicircle (free Gaussian) law if $a = b = 0$; see [Voiculescu, 2000, Section 2.5];
2. the Marchenko-Pastur (free Poisson) type law if $b = 0$ and $a \neq 0$; see [Voiculescu, 2000, Section 2.7];
3. the free Pascal (negative binomial) type law if $b > 0$ and $a^2 > 4b$; see [Saitoh and Yoshida, 2001, Example 3.6];
4. the free Gamma type law if $b > 0$ and $a^2 = 4b$; see [Bożejko and Bryc, 2005, Proposition 3.6];
5. the free analog of hyperbolic type law if $b > 0$ and $a^2 < 4b$; see [Anshelevich, 2003, Theorem 4];
6. the free binomial type law if $-1 \leq b < 0$; see [Saitoh and Yoshida, 2001, Example 3.4].

Morris Thm | Skip Proof

Proof of Theorem 14.

With

$$z = m + \frac{V(m)}{m - m_0}, \tag{13}$$

we showed that $G_\nu(z) = \frac{m-m_0}{V(m)}$. Solving (13) for m we get

$$m = \frac{z - a - \sqrt{(a-z)^2 - 4\,(1+b)}}{2\,(1+b)},$$

and

$$G(z) = \frac{a + z + 2\,b\,z - \sqrt{(a-z)^2 - 4\,(1+b)}}{2\,(1 + az\; + b\,z^2)}. \tag{14}$$

This Cauchy-Stieltjes transform appears in [Anshelevich, 2003, Bożejko and Bryc, 2005, Bryc and Wesołowski, 2005, Saitoh and Yoshida, 2001] and defines the free-Meixner laws. □

Theorem ([Bryc and Ismail, 2005])

Suppose V is analytic in a neighborhood of m_0, $V(m_0) > 0$, and $\lambda > 0$. Then the following conditions are equivalent.

- *$V(\cdot)$ is a variance function of a free exponential family;*
- *There exists a probability measure ν with free cumulants $c_1 = m_0$, and for $n \geq 1$*
$$c_{n+1} = \frac{1}{n!} \frac{d^{n-1}}{dx^{n-1}} (V(x))^n \Big|_{x=m_0} . \tag{15}$$
- *Measure ν is compactly supported and there exists an interval $(A, B) \ni m_0$ such that (5) defines a free exponential family centered at m_0 with variance function V.*

Skip Proof

Proof.

The inverse $K_\nu = G_\nu^{-1}$ is well defined for m close to m_0, and

$$m + \frac{V(m)}{m - m_0} = K_\nu\left(\frac{m - m_0}{V(m)}\right).$$

so the R-transform of ν satisfies

$$R_\nu\left(\frac{m - m_0}{V(m)}\right) = m. \tag{16}$$

There exists $\varepsilon > 0$ such that for $k \geq 1$ we have

$$c_{k+1} = \frac{1}{2\pi i}\oint_{|z|=\varepsilon} \frac{R(z) - m_0}{z^{k+1}}dz.$$

Denote $V_0(z) = V(z + m_0)$. Substituting $z = (\xi - m_0)/V(\xi)$ and changing the path of integration we get $\square$

Proof Cont.

$$c_{k+1} = \frac{1}{2\pi i}\oint_{|\xi - m_0|=\delta} \frac{V^k(\xi)}{(\xi - m_0)^k}\left(1 - \frac{(\xi - m_0)V'(\xi)}{V(\xi)}\right) d\xi$$

$$= \frac{1}{2\pi i}\oint_{|z|=\delta} \frac{V_0^k(z)}{z^k} dz - \frac{1}{2\pi i \lambda^k}\oint_{|z|=\delta} \frac{V_0^{k-1}(z)}{z^{k-1}} V_0'(z) dz.$$

Notice that

$$\frac{d}{dz}\frac{V_0^k(z)}{z^{k-1}} = -(k-1)\frac{V_0^k(z)}{z^k} + k\frac{V_0^{k-1}}{z^{k-1}} V_0'(z).$$

Therefore,

$$\oint_{|z|=\delta} \frac{V_0^{k-1}(z)}{z^{k-1}} V_0'(z) dz = \frac{k-1}{k}\oint_{|z|=\delta} \frac{V_0^k(z)}{z^k} dz.$$

$\square$

Proof Cont.

Thus

$$c_{k+1} = \frac{1}{k}\frac{1}{2\pi i}\oint_{|z|=\delta}\frac{V_0^k(z)}{z^k}dz = \frac{1}{k!}\frac{d^{k-1}}{dz^{k-1}}V_0^k(z)\Big|_{z=0}. \tag{17}$$

Suppose now that a probability measure ν satisfies (15) and $\int x\nu(dx) = m_0$. We first verify that ν has compact support. Since V is analytic, (15) is equivalent to

$$c_{k+1} = \frac{1}{k}\frac{1}{2\pi i}\oint_{|z|=\delta}\frac{V_0^k(z)}{z^k}dz. \tag{18}$$

Thus there exist $M > 0$ such that $|c_k| \leq M^k$. □

Proof Cont.

Denoting by $\mathcal{NC}[n]$ the set of non-crossing partitions of $\{1, 2, \ldots, n\}$, from [Hiai and Petz, 2000, (2.5.8)] we have

$$\int x^{2n}\nu(dx) = \sum_{\mathcal{V}\in\mathcal{NC}[2n]} \prod_{B\in\mathcal{V}} c_{|B|} \leq M^{2n}\#\mathcal{NC}[2n] = M^{2n}\frac{1}{2n+1}\binom{4n}{2n};$$

for the last equality, see [Hiai and Petz, 2000, (2.5.11)]. Thus

$$\limsup_{p\to\infty}\left(\int |x|^p\nu(dx)\right)^{1/p} \leq 2M < \infty,$$

and ν has compact support. (See also [Benaych-Georges, 2004, Theorem 1.3].)
From $\mathrm{supp}(\nu) \subset [-2M, 2M]$ we deduce that the Cauchy-Stieltjes transform $G_\nu(z)$ is analytic for $|z| > 2M$, and the R-series is analytic for all $|z|$ small enough. $\square$

Proof Cont.

Since $V(m_0) \neq 0$ we see that $z \mapsto \frac{z-m_0}{V(z)}$ is invertible in a neighborhood of $z = m_0$. Denoting by h the inverse, we have

$$h\left(\frac{z-m_0}{V(z)}\right) = z.$$

From $c_1(\nu) = m_0$ we see that $R(m_0) = 0 = h(m_0)$. Repeating the reasoning that lead to (15) with function h, we see that all derivatives of h at $z = m_0$ match the derivatives of R. Thus $h(z) = R(z)$ and (16) holds for all m in a neighborhood of m_0. For analytic G_ν, the latter is equivalent to (7) holding for all m close enough to m_0. Thus $V(m)$ is the variance function of a free exponential family generated by ν with $m \in (m_0 - \delta, m_0 + \delta)$ for some $\delta > 0$. $\square$

The λ-fold free convolution $\nu^{\boxplus\lambda}$ is well defined for all $\lambda \geq 1$, see [Nica and Speicher, 1996]. Then measure

$$\nu_\lambda(U) := \nu^{\boxplus\lambda}(\lambda U))$$

is the law of the "sample average" of λ free elements with law ν.

Proposition (B-Ismail (2005))

If ν generates free exponential family centered at m_0 and its variance function V is analytic in a neighborhood of m_0, then for $\lambda \geq 1$, measure ν_λ generates the exponential family centered at m_0 with the variance function $V(m)/\lambda$.
Moreover, if $V(m)/\lambda$ if a variance function of a free exponential family for all $\lambda > 0$, then ν is $\boxplus$-infinitely divisible.s

We note that in contrast to classical natural exponential families, in (5) the interval (A, B) varies with λ, see Example 13.

Classical Version Skip Proof End Here

Proof.

The free cumulants of $\nu_{m_0,\lambda}$ are $c_1(\nu_{m_0,\lambda}) = c_1(\nu_{m_0,\lambda_0}) = m_0$ and for $n \geq 1$

$$c_{n+1}(\nu_{m_0,\lambda}) = \frac{1}{\rho^n} c_{n+1}(\nu_{m_0,\lambda_0}) = \frac{1}{\rho^n \lambda_0^n n!} \frac{d^{n-1}}{dx^{n-1}} (V(x))^n \Big|_{x=m_0}$$
$$= \frac{1}{\lambda^n n!} \frac{d^{n-1}}{dx^{n-1}} (V(x))^n \Big|_{x=m_0}.$$

Theorem 15 implies that V/λ is the variance function of the free exponential family generated by ν_λ and centered at m_0.
If $\nu_{m_0,1/n}$ exists for all $n \in \mathbb{N}$, then the previous reasoning together with uniqueness theorem (Proposition 12) implies that $\nu = (D_n(\nu_{m_0,1/n}))^{\boxplus n}$, proving $\boxplus$-infinite divisibility. $\square$

The End

Given $k(x, \theta)$.

Definition ([Wesołowski, 1999])

The kernel family $\mathcal{K}$ consists of probability measures

$$\left\{ \frac{k(x,\theta)}{M(\theta)} \nu(dx) : \ \theta \in \Theta \right\},$$

where $M(\theta) = \int k(x,\theta)\nu(dx)$ is the normalizing constant.

- Natural exponential family: $k(x, \theta) = \exp(\theta(x - m_0))$, where auxiliary parameter m_0 cancels out.
- Free exponential families: $k(x, \theta) = \frac{1}{1-\theta(x-m_0)}$.

Suppose ν is a compactly supported with $\int x d\nu = m_0$. Then

$$M(\theta) = \int \frac{1}{1-\theta(x-m_0)} \nu(dx).$$

The kernel family for $k = \frac{1}{1-\theta(x-m_0)}$ is the family of probability measures

$$\mathcal{K}(\nu;\Theta) = \left\{ P_\theta(dx) = \frac{1}{M(\theta)(1-\theta(x-m_0))} \nu(dx) : \theta \in \Theta \right\}, \tag{19}$$

where Θ is an open set on which $M(\theta)$ is well defined. (One can take $\Theta = (-\varepsilon, \varepsilon)$ with $\varepsilon > 0$ small enough.)

Theorem

Every compactly supported measure ν generates a free exponential family.

- *There exists a function V which is positive and analytic in the neighborhood of $m_0 = \int x\nu(dx)$, and an interval $(A, B) \ni m_0$ such that V is the variance function of a free exponential family $\mathcal{F}_{m_0}(V)$ with the generating measure ν.*
- *Furthermore, $\mathcal{F}(V) = \mathcal{K}(\nu; \Theta)$ for some open set $\Theta \subset \mathbb{R}$.*

Proof.

Without loss of generality we take $m_0 = 0$. From

$$\mathcal{K}(\nu;\Theta) = \left\{ P_\theta(dx) = \frac{1}{M(\theta)(1-\theta(x-m_0))}\nu(dx) : \theta \in \Theta \right\}$$

we compute $m(0) = \int x\nu(dx) = 0$ and more generally

$$m(\theta) = \int xP_\theta(dx) = \frac{M(\theta)-1}{\theta M(\theta)}. \tag{20}$$

Since $M(\theta)$ is analytic at $\theta = 0$ and $M(0) = 1$, we see that $m(\theta)$ is analytic for $|\theta|$ small enough. Furthermore,

$$m'(\theta) = \int \frac{x^2}{(1-x\theta)^2}\nu(dx) > 0$$

for all $|\theta|$ small enough. Thus $\theta \mapsto m(\theta)$ is invertible in a neighborhood of 0; let ψ be the inverse function. $\square$

Proof Cont.

Note that if $G_\nu(z)$ is the Cauchy-Stieltjes transform, then with $z = 1/\theta$ we have $G_\nu(z) = \theta M(\theta)$. Thus (20) is equivalent to

$$\frac{1}{\theta} - m(\theta) = \frac{1}{G_\nu(z)}. \tag{21}$$

We now calculate the variance $v(\theta) = \int x^2 P_\theta(dx) - m^2(\theta)$. Since

$$\int x^2 P_\theta(dx) = \int \frac{x^2 - x/\theta + x/\theta}{M(\theta)(1-\theta x)} \nu(dx) = \frac{m(\theta)}{\theta}$$

we see that the variance is

$$v(\theta) = m(\theta) \left(\frac{1}{\theta} - m(\theta) \right). \tag{22}$$

□

Proof Cont.

Let $V(m) = v(\psi(m))$ denote the variance function in parametrization of (a subset of) $\mathcal{K}$ by the mean; clearly V is an analytic function. With $z = 1/\psi(m)$ combining (22) with (21) we get

$$\frac{m}{V(m)} = G_\nu(z).$$

Therefore, from (22) we see that

$$R_\nu\left(\frac{m}{V(m)}\right) = \frac{1}{\theta} - \frac{V(m)}{m} = m.$$

Since R_ν is analytic and we established (16), from the first part of proof of Theorem 15 we get (15), and from the second part we deduce that V is a variance function of the free exponential family generated by ν.

$\square$

Proof Cont.

It is clear that the families $\mathcal{K}(\nu;\Theta)$ defined by (19) with $\Theta = \psi^{-1}(-\delta,\delta)$, and $\mathcal{F}(V)$ defined by (5) with the interval $(A,B) = (-\delta,\delta)$ coincide. $\square$

The end

Thank You

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The End

Proof of Proposition 9.

Differentiating we get

$$\frac{\partial}{\partial m} w_{m,x} = \frac{\partial}{\partial m} e^{\psi(m)x - \kappa(\psi(m))}$$
$$= \psi'(m)(x - \kappa'(\psi(m))) \exp\left(\psi(m)x - \kappa(\psi(m))\right).$$

As $\kappa'(\psi(m)) = m$ and $\psi'(m) = 1/\kappa''(\psi(m)) = 1/V(m)$, (3) follows. □

◂ QED

Proof of Remark ??.

To prove infinite divisibility, without loss of generality we may concentrate on fixed $W_1(m_0, dx) \in \mathcal{F}(\nu)$.

$$\mathcal{F}(\nu) = \mathcal{F}(W_1(m_0, dx))$$

For $\lambda = 1/k$ where $k = 1, 2, \dots,$ let $W_\lambda(m, dx), m \in (A, B)$ be the solution of (3). The variance function is $V(m)/\lambda = kV(m)$. Denote by ν the dilation of measure $W_\lambda(m_0, dx)$ by k. By (2), the exponential family $\mathcal{F}(\nu^{*k})$ has the same variance function $V(m)$ as the exponential family $\mathcal{F}(W_1(m_0, dx))$. By uniqueness of parametrization by the means, $W_1(m_0, dx) = \nu^{*k}(dx)$, so infinite divisibility follows. $\square$

QED