COMPUTING MOMENTS OF FREE ADDITIVE CONVOLUTION OF MEASURES

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Abstract. This short note explains how to use ready-to-use components of symbolic software to convert between the free cumulants and the moments of measures without sophisticated programming. This allows quick access to low order moments of free convolutions of measures, which can be used to test whether a given probability measure is a free convolution of other measures.

1. Introduction

Free additive convolution $\mu \boxplus \nu$ of compactly supported probability measures μ, ν was introduced by Voiculescu [11]; it was extended to measures with finite variance in [6] and to arbitrary probability measures in [1]. It becomes an increasingly important tool in applications, see [10].

There are two definitions of free convolution which offer different advantages. The analytical definition relies on the Cauchy-Stieltjes transforms and employs inverse functions which may be difficult to implement on a computer; but this definition is applicable to arbitrary probability measures [1, 6]. Ref. [7] advances this approach towards computer-assisted computations when the Cauchy-Stieltjes transforms of the distributions satisfy polynomial equations.

The combinatorial definition of Speicher [8] is applicable only to probability measures with all moments, but this case is often encountered in practice, and it yields direct analytical relations between polynomials that can be programmed into symbolic software. This definition relies on free cumulants and their combinatorial relation to moments, through the sums over the latticed of non-crossing partitions. Our goal is to express this elegant theory in the analytic form which is ready for use with symbolic software. The resulting formulas can be used to explore whether a given probability measure with known moments can be represented as a free convolution of some other measures with known moments. In fact, Theorem 1 originated in the early stages of work on [3, Theorem 1.3] as an exploratory tool to identify the limiting law.

2. Background on free convolution

2.1. Free cumulants.

For a probability measure μ which has finite moments

$$m_k = \int x^k \mu(dx)$$

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of all orders $k = 1, 2, \dots$, let

$$M(z) = 1 + \sum_{k=1}^{\infty} m_k z^k \tag{1}$$

be the formal moment generating function. Define the R-series as the formal power series

$$R(z) = \sum_{k=1}^{\infty} c_k z^{k-1} \tag{2}$$

such that

$$M(z) = 1 + zM(z)R(zM(z)) \tag{3}$$

see [9, formula (75)]. (We note that this composition of formal power series is indeed well defined.) The coefficients $c_k = c_k(\mu)$ in (2) are called free cumulants of probability measure μ.

Denoting by $R_\mu(z)$ the R-series for probability measure μ, the fundamental result of the theory is that for a pair of probability measures μ, ν with finite moments there exists a probability measure $\mu \boxplus \nu$ called the free (additive) convolution of μ, ν such that

$$R_{\mu\boxplus\nu}(z) = R_\mu(z) + R_\nu(z).$$

This relation determines uniquely probability measure $\mu \boxplus \nu$ when it is uniquely determined by moments; this is the case, for example, when measures μ, ν have compact support. Of course, the equivalent form of the defining relation is

$$c_k(\mu \boxplus \nu) = c_k(\mu) + c_k(\nu), \ k = 1, 2, \dots.$$

2.2. **Algorithmic version of the relations.** For computer usage, the relation between moments and free cumulants should be expressed in terms of the polynomials obtained by truncation of the formal power series. Let

$$M_n(z) \equiv M(z) \mod z^{n+1}, \ R_n(z) \equiv R(z) \mod z^{n+1}$$

denote the n-th truncations of the formal series, i.e., $M_n(z) = 1 + \sum_{k=1}^{n} m_k z^k$ and $R_n(z) = \sum_{k=1}^{n+1} c_k z^{k-1}$.

Theorem 1. (i) *The consecutive truncations of $M(z)$ are determined from the consecutive truncations of $R(z)$ by the following recurrence. With $M_0(z) = 1$,*

$$M_n(z) \equiv 1 + zM_{n-1}(z)R_{n-1}(zM_{n-1}(z)) \mod z^{n+1}, \ n \geq 1. \tag{4}$$

(ii) *Coefficients c_k in (2) are determined from the consecutive truncations M_n of $M(z)$ by $c_1 = M_1'(0)$, and for $2 \leq k \leq n$,*

$$c_k = -\frac{1}{k-1}\frac{1}{k!}\frac{d^k}{dz^k}\left.\frac{1}{M_n^{k-1}(z)}\right|_{z=0}. \tag{5}$$

Remark 1. One can also write (4) as

$$M_n(z) \equiv 1 + \sum_{k=1}^{n} m_{k-1} z^k R_{n-k-1}(zM_{n-k-1}(z)) \mod z^{n+1}.$$

Proof. Since $(zM(z))^k \equiv 0 \mod z^{n+1}$ for $k \geq n+1$, we get $M_n(z) \equiv 1 + zM_{n-1}(z)R_{n-1}(zM(z)) \mod z^{n+1}$. Considering separately $k=0$ and $k>0$ we see that for all $k \geq 0$

$$(zM(z))^k \equiv z^k M_{n-k}^k(z) \equiv z^k M_{n-1}^k(z) \mod z^{n+1}.$$

Thus $R_{n-1}(zM(z)) \equiv R_{n-1}(zM_{n-1}(z)) \mod z^{n+1}$.

We now prove (5). For $n \geq k \geq 1$ we have

$$c_k = \frac{1}{2\pi i} \oint_{|u|=\varepsilon} \frac{1+uR_n(u)}{u^{k+1}} du.$$

Since $z \mapsto zM_n(z)$ maps the origin back into itself, and the derivative at 0 is $1 \neq 0$, for small enough $\varepsilon > 0$ we can substitute $u = zM_n(z)$ to get

$$c_k = \frac{1}{2\pi i} \oint_{|z|=\varepsilon} \frac{1+zM_n(z)R_n(zM_n(z))}{z^{k+1}M_n^{k+1}(z)} (M_n(z) + zM_n'(z))dz.$$

From (4) it follows that there are real coefficients $\{d_j\}$ such that $1+zM_n(z)R_n(zM_n(z)) = M_{n+1}(z) + \sum_{j=n+2}^{n^2+n+2} d_j z^j = M_n(z) + \sum_{j=n+1}^{n^2+n+2} d_j z^j$. Since $k \leq n$ and $M_n(z) \neq 0$ in the neighborhood of 0,

$$\frac{(M_n(z) + zM_n'(z)) \sum_{j=n+1}^{n^2+n+2} d_j z^{j-k-1}}{M_n^{k+1}(z)}$$

is an analytic function. So for small enough $\varepsilon > 0$ we have

$$\begin{aligned} c_k &= \frac{1}{2\pi i} \oint_{|z|=\varepsilon} \frac{M_n(z)}{z^{k+1}M_n^{k+1}(z)} (M_n(z) + zM_n'(z))dz \\ &= \frac{1}{2\pi i} \oint_{|z|=\varepsilon} \frac{1}{z^{k+1}M_n^{k-1}(z)} dz + \frac{1}{2\pi i} \oint_{|z|=\varepsilon} \frac{M_n'(z)}{z^k M_n^k(z)} dz. \end{aligned}$$

We now use the fact that the derivative of

$$\frac{-1}{z^k M_n^{k-1}(z)}$$

is

$$\frac{k}{z^{k+1}M_n^{k-1}(z)} + \frac{(k-1)M_n'(z)}{z^k M_n^k(z)}.$$

Thus, for $k > 1$,

$$\oint_{|z|=\varepsilon} \frac{M_n'(z)}{z^k M_n^k(z)} dz = -\frac{k}{k-1} \oint_{|z|=\varepsilon} \frac{1}{z^{k+1}M_n^{k-1}(z)} dz,$$

and

$$c_k = -\frac{1}{(k-1)2\pi i} \oint_{|z|=\varepsilon} \frac{1}{z^{k+1}M_n^{k-1}(z)} dz,$$

which ends the proof. □

3. Applications

In this section we show how to use formulas (4) and (5) to compute free cumulants and moments of free convolutions of measures.

3.1. **Symbolic software implementation.** Using symbolic software and $M_6(z) = 1 + \sum_{k=1}^{6} m_k z^k$, we can use Theorem 1(ii) together with Mathematica code

```
Table[{k, -1/(k-1)/k!D[(M[z])^(1-k),{z,k}]/.{z->0}}, {k,2,6}]
```

to generate the following expressions for the free cumulants.

Table 1. Free cumulants expressed through moments.

$$\begin{aligned}
c_2 &= -{m_1}^2 + m_2 \\
c_3 &= 2\,{m_1}^3 - 3\,m_1\,m_2 + m_3 \\
c_4 &= -5\,{m_1}^4 + 10\,{m_1}^2\,m_2 - 2\,{m_2}^2 - 4\,m_1\,m_3 + m_4 \\
c_5 &= 14\,{m_1}^5 - 35\,{m_1}^3\,m_2 + 15\,{m_1}^2\,m_3 - 5\,m_2\,m_3 + 5\,m_1\,\left(3\,{m_2}^2 - m_4\right) + m_5 \\
c_6 &= -42\,{m_1}^6 + 126\,{m_1}^4\,m_2 + 7\,{m_2}^3 - 56\,{m_1}^3\,m_3 - 3\,{m_3}^2 - 6\,m_2\,m_4 \\
&\quad +21\,{m_1}^2\,\left(-4\,{m_2}^2 + m_4\right) + 6\,m_1\,\left(7\,m_2\,m_3 - m_5\right) + m_6
\end{aligned}$$

Similarly, Theorem1(i) has straightforward implementation in Mathematica:

```
M[z_] = 1; Do[
  Rtmp[z_] = PolynomialMod[z R[z], z^(k + 1)];
  M[z_] = PolynomialMod[1 + Rtmp[z M[z]], z^(k + 1)].
  , {k, 0, 5}]
```

This gives

$$M_5[z] = 1 + zc_1 + z^2\left(c_1{}^2 + c_2\right) + z^3\left(c_1{}^3 + 3c_1c_2 + c_3\right) + z^4\left(c_1{}^4 + 6c_1{}^2c_2 + 2c_2{}^2 + 4c_1c_3 + c_4\right) + z^5\left(c_1{}^5 + 10c_1{}^3c_2 + 10c_1{}^2c_3 + 10c_1c_2{}^2 + 5c_2c_3 + 5c_1c_4 + c_5\right),$$

from which we can read out explicit expressions for low order moments (1). Of course, explicit relations between free cumulants and moments are known in terms of sums over non-crossing partitions [8, 9]; our point here is that these relations have simple implementation in symbolic software.

3.2. **Free cumulants of some classical laws.** Lehner [5, Theorem 4.1] expresses free cumulants as the sum of products of classical cumulants over all connected partitions. Here, we list free cumulants derived from (4) using the well known formulas for moment generating functions $M(z)$ of classical laws from Table 2. Table 3 lists numerical values. We remark that free cumulants c_k of the standard normal law are the number of connecting pairings ([2]) and free cumulants of the Poisson law are the number of connected partitions ([5]) of $\{1, \ldots, k\}$, so Table 3 enumerates these sets for $k \leq 15$. Table 4 expresses free cumulants in terms of the parameters of Poisson and Binomial laws.

Table 2. Notation for some classical laws

Name	Parameters	Distribution	Notation
Poisson	$\lambda > 0$	$e^{-\lambda}\lambda^k/k!, k = 0, 1, 2, \ldots$	$Poiss(\lambda)$
Exponential	$\lambda > 0$	$f(x) = \lambda\exp(-\lambda x)$, $x > 0$	$Exp(\lambda)$
Normal	$\sigma > 0$, m	$\exp(-(x-m)^2/(2\sigma^2))/(\sigma\sqrt{2\pi})$	$N(m, \sigma)$
Binomial	$n \geq 0$, $0 \leq p \leq 1$	$\binom{n}{k}p^kq^{n-k}$, $k = 0, 1, \ldots, n$; $q = 1 - p$.	$Bin(n, p)$
Uniform	$a < b$	$f(x) = (b-a)^{-1}1_{a<x<b}$	$U(a, b)$

Table 3. Free cumulants of some classical laws.

k	$N(0,1)$	Exp(1)	Poiss(1)	$U(-1,1)$
2	1	1	1	$\frac{1}{3}$
3	0	2	1	0
4	1	7	2	$-\frac{1}{45}$
5	0	34	6	0
6	4	206	21	$\frac{2}{945}$
7	0	1476	85	0
8	27	12123	385	$-\frac{1}{4725}$
9	0	111866	1907	0
10	248	1143554	10205	$\frac{2}{93555}$
11	0	12816572	58455	0
12	2830	156217782	355884	$-\frac{1382}{638512875}$
13	0	2057246164	2290536	0
14	38232	29111150620	15518391	$\frac{4}{18243225}$
15	0	440565923336	110283179	0

Table 4. Free cumulants of the general Poisson Binomial laws.

k	$Poiss(\lambda)$	$Bin(n,p)$
2	λ	npq
3	λ	$np\,(q-p)\,q$
4	$\lambda\,(1+\lambda)$	$npq\,(1+(-6+n)\,pq)$
5	$\lambda\,(1+5\lambda)$	$np\,(q-p)\,q\,(1+(-12+5n)\,pq)$
6	$\lambda\,(1+4\lambda\,(4+\lambda))$	$npq\left(1+2pq\left(-15(q-p)^2+n\,(8+(-41+2n)\,pq)\right)\right)$
7	$\lambda\,(1+42\lambda\,(1+\lambda))$	$np\,(q-p)\,q\,(1+6pq\,(-10+7n+(60+7\,(-8+n)\,n)\,pq))$

3.3. **Free convolutions with the semicircle and Marchenko-Pastur laws.** Free convolutions with the semicircle and Marchenko-Pastur laws arise frequently as the asymptotic spectra of sums of independent matrices, see [4]. The free cumulants of the semicircle law ω_σ are zero except for $c_2 = \sigma^2$; the free cumulants of the Marchenko-Pastur law π_λ are all equal to $\lambda > 0$. Therefore, it is easy to describe how the free cumulants change, and then to compute the moments from (4). To compute the moments of free convolutions with a semicircle law ω_σ, we change the value of the second cumulant c_2 in the results of the previous section to $c_2+\sigma^2$, and then use (4) to derive the corresponding moments. Similarly, to compute moments of free convolution with Marchenko-Pastur law, we replace the k-th free cumulant c_k with $c_k + \lambda$ and apply (4).

To illustrate this method, we apply it first to the general relations (Table 1) between moments and free cumulants. We get the following relations.

TABLE 5. Moments $M_k = \int x^k(\mu \boxplus \omega_\sigma)(dx)$ expressed in terms of $m_k = \int x^k\mu(dx)$.

k	M_k
1	m_1
2	$\sigma^2 + m_2$
3	$3\sigma^2 m_1 + m_3$
4	$2\sigma^4 + 2\sigma^2 {m_1}^2 + 4\sigma^2 m_2 + m_4$
5	$5\sigma^2 m_1\left(2\sigma^2 + m_2\right) + 5\sigma^2 m_3 + m_5$
6	$5\sigma^6 + 15\sigma^4 {m_1}^2 + 15\sigma^4 m_2 + 3\sigma^2 {m_2}^2 + 6\sigma^2 m_1 m_3 + 6\sigma^2 m_4 + m_6$

TABLE 6. Moments $M_k = \int x^k(\mu \boxplus \pi_\lambda)(dx)$ expressed in terms of $m_k = \int x^k\mu(dx)$.

k	M_k
1	$\lambda + m_1$
2	$\lambda + \lambda^2 + 2\lambda m_1 + m_2$
3	$\lambda + 3\lambda^2 + \lambda^3 + 3\lambda\left(1+\lambda\right) m_1 + 3\lambda m_2 + m_3$
4	$\lambda + 6\lambda^2 + 6\lambda^3 + \lambda^4 + 4\lambda\left(1 + 3\lambda + \lambda^2\right) m_1 + 2\lambda {m_1}^2 + 2\lambda\left(2+3\lambda\right) m_2 + 4\lambda m_3 + m_4$
5	$\lambda + 10\lambda^2 + 20\lambda^3 + 10\lambda^4 + \lambda^5 + 5\lambda\left(1+2\lambda\right) {m_1}^2 + 5\lambda\left(1 + 4\lambda + 2\lambda^2\right) m_2$
	$+5\lambda m_1\left(1 + 6\lambda + 6\lambda^2 + \lambda^3 + m_2\right) + 5\lambda m_3 + 10\lambda^2 m_3 + 5\lambda m_4 + m_5$

As a further illustration of the method, we give the following tables of low order moments which are based on Table 3.

Table 7. Moments of order $k \leq 10$ of free convolutions with the semicircle law

k	$N(0,1) \boxplus \omega_1$	$Exp(1) \boxplus \omega_1$	$Poiss(1) \boxplus \omega_1$
1	0	1	1
2	2	3	3
3	0	9	8
4	9	36	27
5	0	170	97
6	56	962	385
7	0	6384	1647
8	431	48954	7598
9	0	426666	37608
10	3942	4165692	199217

Table 8. Moments of order $k \leq 10$ of free convolutions with the Marchenko-Pastur law

k	$N(0,1) \boxplus \pi_1$	$Exp(1) \boxplus \pi_1$	$Poiss(1) \boxplus \pi_1$
1	1	2	2
2	3	6	6
3	8	23	22
4	27	104	91
5	92	537	409
6	339	3134	1958
7	1276	20659	9874
8	4985	154044	52134
9	19841	1297982	287333
10	80801	12293798	1651337

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