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Probability Theory and Related Fields

Remarks on Properties of Probability Distributions Determined by Conditional Moments

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Summary. In this paper we consider some properties of rotation – invariant distributions on R^n, which are determined by a form of conditional moment of order $\alpha > 0$. In particular we prove that the Gaussian distribution is determined uniquely by its conditional moments and we investigate the related question of finiteness of exponential moments. The case of general $\alpha > 0$ appears to be more difficult to analyze than the case $\alpha = 2$, studied previously by other authors.

0. Introduction

Characterizations of probability distributions by their properties attracted attention because of practical importance to recognize a class of distributions before any statistical inference is made. A classical result rarely mentioned in this context is a well-known characterization by P. Lévy of diffusion processes as processes with continuous trajectories satisfying some constraints on the form of the first two conditional moments (see Billingsley, (1968) Theorem 19.3). In recent years there has been an increasing number of examples of those situations, in which knowing the form of some conditional moments provides unexpectedly accurate information about distributions of dependent sequences. This phenomenon is peculiar to dependent situations and so far there is no general theory of it. Most of the known situations deal with Gaussian processes under various assumptions: continuity of trajectories, see Billingsley (1968) Theorem 19.3, L_2-continuity, see Plucińska (1983), Wesołowski (1984), L_2-differentiability, see Bryc, Szabłowski (1984), discrete time, see Bryc, Plucińska (1985). There is also unexpected information about integrability of random variables hidden in integrability of some conditional moments, see Bryc (1985).

Recently it was checked, that the Poisson process is uniquely determined by a form of its conditional moments of first two orders, see Bryc (1987). In a series of papers, P. Szabłowski (1986 a–c, 1987) considered those properties of the so-called elliptically contoured distributions, which are determined by conditional variances and conditional covariances, see also Cambanis et al.

(1981). It appears that in general, finite dimensional distributions of L_2-differentiable processes frequently are determined uniquely by their first two conditional moments, see Szabłowski (1986c).

Most of that research concentrated on the role played by the first two conditional moments (with some exceptions: Bryc, Szabłowski (1984) considered conditional expectations of Hermite polynomials, Cambanis et al. (1981) characterized Gaussian distribution by conditional moment of any integer order).

The subject of this paper is to consider some properties of elliptically contoured distributions determined by a conditional moment of arbitrary order $\alpha>0$. This was suggested by Szabłowski (1987) Remark 4, and Cambanis et al. (1981) Corollary 8 and their remark thereafter. It appears that the case $\alpha\neq 2$ is considerably more difficult and, except of the Gaussian case, more complicated.

Elliptically contoured distributions are affine transformations of rotation invariant distributions. Therefore in statements of results we shall limit our attention to rotation-invariant distributions, which we shall call here "spherically invariant". Generalization of results to elliptically contoured distribution would just complicate notations.

In Sect. 1 we shall show that among spherically invariant distributions both the Gaussian distribution and the uniform distribution on a sphere (the latter under an additional restriction on the dimension of variables) are uniquely determined by a form of conditional moment of order $\alpha>0$. We shall also provide a simple proof of (or a version of) Szablowski (1986a) which says that, under mild technical assumption, a form of conditional moment of order $\alpha=2$ determines uniquely any spherically invariant distribution.

In Sect. 2 we shall study the finiteness of exponential moments. In some sense the results generalize part of Szabłowski (1987) Theorem 2 even in the case $\alpha=2$, and Theorem 2.1 below permits to show $E\exp(\varepsilon X_1^2)<\infty$ in some situations with $\alpha<2$. However it is more difficult to analyze by our method those situations, when the conditional moment is not bounded; the case $\alpha=2$ is well covered by Szabłowski.

In this paper we shall use the standard notation $\Gamma(x)=\int_0^\infty t^{x-1}e^{-t}\,dt$, $B(x,y)=\Gamma(x)\Gamma(y)/\Gamma(x+y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$, $\operatorname{Re}x, \operatorname{Re}y>0$. Equalities between random variables are always interpreted as equality almost everywhere.

1. Uniqueness

Let $n\geqq 2$ be fixed. Recall, that $\mathbf{X}=(X_1,\ldots,X_n)$ has a spherically invariant distribution, if for each $\alpha_1,\ldots,\alpha_n\in R$ such that $\sum\alpha_k^2=1$ we have $\sum\alpha_k X_k\simeq X_1$ (here and below $\simeq$ denotes equality of distributions).

Let $1\leqq m<n$ be fixed. Denote $\mathbf{X}_1=(X_1,\ldots,X_m)$, $\mathbf{X}_2=(X_{m+1},\ldots,X_n)$, $R=(\sum X_k^2)^{1/2}=\|\mathbf{X}\|$. It is known, that the distribution of $\mathbf{X}$ is determined uniquely by one of the following: distribution of X_1, distribution of $\|\mathbf{X}_2\|$, distribution

of R, see Cambanis et al. (1981). Also it is known, that $E(\|\mathbf{X}_1\|^\alpha|\mathbf{X}_2) = E(\|\mathbf{X}_1\|^\alpha|\,\|\mathbf{X}_2\|)$ a.e. (whenever defined).

The following result result generalizes Cambanis et al. (1981) Corollary 8 a to arbitrary exponents $\alpha > 0$.

Theorem 1.1. *If* $\mathbf{X}$ *has a spherically invariant distribution such that for some* $\alpha > 0$ $E|X_1|^\alpha < \infty$ *and* $E(\|\mathbf{X}_1\|^\alpha|\mathbf{X}_2) = \text{const}$ a.e. *then* $\mathbf{X}$ *is a Gaussian vector.*

Remark. Theorem 1.1 was proved by Cambanis et al. (1981) for $\alpha = 1, 2, \ldots$. Their method does not generalize to noninteger α. A related problem was also considered by Richards (1984).

Our method of proof of Theorem 1.1 will also provide easy access to the following version of a theorem due to Szabłowski (1986 a), see also Galambos, Kotz (1978), Theorem 2.3.2.

Theorem 1.2. *Suppose* $\mathbf{X}$ *has spherically invariant distribution such that* $E|X_1|^2 < \infty$ *and* $P(\|\mathbf{X}\| = 0) = 0$. *Let the function* $c(r^2) := E(\|\mathbf{X}_1\|^2 |\, \|\mathbf{X}_2\| = r)$ *be such that* $\int_0^a 1/c(x)\,dx < \infty$ *for each* a *with* $0 < a < \inf\{x\colon c(x) = 0\}$. *Then the distribution of* $\mathbf{X}$ *is uniquely determined by the function* $c(r)$, $r \geqq 0$.

The following lemma reduces proofs of Theorems 1.1, 1.2 to investigation of solutions of an integral equation.

Lemma 1.4. *Suppose* $\mathbf{X}$ *has a spherically invariant distribution and* $P(\mathbf{X} = 0) = 0$. *Then random variable* $\|\mathbf{X}_2\|$ *has a density* f *with respect to Lebesgue measure on* R.

Suppose furthermore that for some $\alpha > 0$, $E|X_1|^\alpha < \infty$ *and let* $c_\alpha(x)$, $x \geqq 0$ *be a function such that*

$$E(\|\mathbf{X}_1\|^\alpha|\mathbf{X}_2) = c_\alpha(\|\mathbf{X}_2\|^2).$$

Then the function $\phi(x) = x^{(m+1-n)/2} f(x^{1/2})$ *satisfies the integral equation*

$$c_\alpha(x)\,\phi(x) = 1/B(\alpha/2, m/2) \int_x^\infty (y-x)^{\alpha/2-1}\,\phi(y)\,dy, \qquad x \geqq 0. \tag{1.1}$$

Proof. Let H be a distribution of $\|\mathbf{X}\|$. It is known (see Cambanis et al. (1981) formula (8) and the proof of their Corollary 8 a), that $\|\mathbf{X}_2\|$ has the probability density function f which satisfies

$$f(x) = C x^{n-m-1} \int_x^\infty r^{-(n-2)} (r^2 - x^2)^{m/2-1}\,dH(r), \qquad x \geqq 0 \tag{1.2}$$

(the integral being finite except of a set of x's of R^1-Lebesgue measure 0; $C = 2\Gamma(n/2)\Big/\left(\Gamma(m/2)\,\Gamma\left(\frac{n-m}{2}\right)\right)$ is a norming constant). Also

$$c_\alpha(x^2)\,f(x) = C x^{n-m-1} \int_x^\infty (r^2 - x^2)^{(m+\alpha)/2-1}\,r^{-(n-2)}\,dH(r), \qquad x \geqq 0. \tag{1.3}$$

Since $(r^2-x^2)^{(m+\alpha)/2-1}=2/B(\alpha/2, m/2)\int_x^r (t^2-x^2)^{\alpha/2-1}(r^2-t^2)^{m/2-1}\,t\,dt$, (1.1) follows from (1.2) and (1.3) by Fubini theorem and change of variable from x to $x^{1/2}$.

Proof of Theorem 1.1. First observe, that without losing generality we may assume $P(\mathbf{X}=0)=0$. Indeed if $P(\mathbf{X}=0)>0$, then $P(E(\|\mathbf{X}_1\|^\alpha|\mathbf{X}_2)=0)>0$ and since $c_\alpha(x)=$const, therefore $c_\alpha(x)\equiv 0$. This in turn implies $E\|\mathbf{X}_1\|^\alpha=0$. Thus, $\mathbf{X}_1=0$ and $\mathbf{X}=0$ and if $P(\mathbf{X}=0)>0$, the theorem is proved.

Assume $P(\mathbf{X}=0)=0$. Then by Lemma 1.4 the density f of $\|\mathbf{X}_2\|$ exists and to prove the theorem it suffices to show that the integral Eq. (1.1) has a unique solution in the class of all measurable functions ϕ such that $\phi(x)\geqq 0$, and $\int_0^\infty x^{n-m-1}\,\phi(x^2)\,dx=1$.

Indeed uniqueness of the solution implies that $\phi(x)$ and thus the density $f(x)$ of $\|\mathbf{X}_2\|$ is as in the Gaussian case. This implies that $\mathbf{X}$ is Gaussian (see Cambanis et al. (1981) for formulas relating distribution of $\mathbf{X}$ to a density function of $\|\mathbf{X}_2\|$.).

To prove the theorem it remains therefore to show that the integral equation

$$\phi(x)=K\int_x^\infty (y-x)^{\beta-1}\,\phi(y)\,dy \tag{1.4}$$

where $\beta=\alpha/2$, $K=(B(\beta, m/2)\,C_\alpha)^{-1}$, has a unique solution under the constraints (i) $\phi(x)\geqq 0$ for each $x\geqq 0$; (ii) $\int_0^\infty x^{(n-m)/2-1}\,\phi(x)\,dx=2$.

Let $\tilde{\phi}(s)=\int_0^\infty x^{s-1}\,\phi(x)\,dx$ be the Mellin transform of ϕ. From Theorem 2.2 below it follows that $\tilde{\phi}(s)$ is well defined for each complex s with $\operatorname{Re} s>(n-m)/2$. Applying the Mellin transform to both sides of (1.4) and switching the order of integrals on the right hand side (which is permitted because the integrands are non-negative) we obtain for $\operatorname{Re} s>(n-m)/2$

$$\tilde{\phi}(s)=K\,\frac{\Gamma(\beta)\,\Gamma(s)}{\Gamma(\beta+s)}\,\tilde{\phi}(\beta+s). \tag{1.5}$$

Thus the Mellin transform $\tilde{\phi}_1$ of the function $\phi_1(x)=\phi([K\Gamma(\beta)]^{-1/\beta}x)$ has the form

$$\tilde{\phi}_1(s)=\Gamma(s)\,p(s) \tag{1.6}$$

where $p(s)$ is a periodic function with period equal β, and $\operatorname{Re} s>(n-m)/2$. Moreover, since $\tilde{\phi}_1(s)$ is analytic in the half-plane $\operatorname{Re} s>(n-m)/2$ and $\Gamma(s)\neq 0$ for $\operatorname{Re} s>0$, therefore $p(s)$ is analytic (i.e., has an analytic extension to C). Also $p(t)\in R$ for any real t.

We shall show that $p(s)=$const. Indeed, since $\phi(x)\geqq 0$ it can be easily checked (either directly, or using the well known relation between Mellin and Fourier transforms, see e.g., Titchmarsh (1937), p. 8), that $\tilde{\phi}_1(s)$ is a completely monotonic function and in particular (see e.g., Akhiezer (1965), p. 210) $\tilde{\phi}_1(t)$ is logarithmically convex for all large enough real t. This implies that for all large enough real t

$$\frac{d^2}{dt^2}\ln p(t)+\frac{d^2}{dt^2}\ln \Gamma(t)\geqq 0.$$

Moreover, it is known (see e.g., Magnus, Oberhettinger (1949), p. 3) that

$$\frac{d^2}{dt^2}\ln \Gamma(t)=\sum_{n=0}^{\infty}\frac{1}{(t+n)^2}\leqq\sum_{n=0}^{N}\frac{1}{(t+n)^2}+\frac{1}{N}\text{ for any } N\geqq 1, t>0.$$

Therefore $\lim\limits_{t\to\infty}\frac{d^2}{dt^2}\ln \Gamma(\mathrm{t})=0$ and thus $\frac{d^2}{dt^2}\ln p(t)\geqq 0$ for all $t>0$. This means that $\frac{d}{dt}\ln p(t)$ is a continuous, periodic and non-decreasing function. Hence $\frac{d}{dt}\ln p(t)=\text{const}$ and since $p(t)\in R$ this proves that $p(t)=\text{const}$, i.e., by the uniqueness of analytic continuation $p(s)=\text{const}$ for all $s\in C$.

Thus, we proved that $\tilde{\phi}_1(s)=\text{const}\,\Gamma(s)$ for all $s\in C$ such that $\operatorname{Re} s>(n-m)/2$ and inverting the Mellin transform $\phi(x)=\text{const}\exp(-xC)$. (Note that const is a norming constant determined uniquelly and $C=\Gamma(m/2+\beta)/(\Gamma(m/2)\,C_\alpha)$. Since the same reasoning applies to any spherically symmetric Gaussian vector, this proves that the distribution of $\|\mathbf{X}_2\|$ is the same as in the Gaussian case and thus $\mathbf{X}$ has a Gaussian distribution. The proof of Theorem 1.1 is completed.

Note. Probabilistic solutions of Eq. (1.4) and more general integral equations were analyzed by J. Deny, see e.g., Ramachandran et al. (1984).

Proof of Theorem 1.2. By Lemma 1.4 it suffices to show that the Eq. (1.1) has unique probabilistic solution. Let $A\leqq\infty$ be smallest number a such that $c(a)=0$. Since $\phi(y)\geqq 0$, it follows from (1.1) that $\phi(x)\equiv 0$ for almost all $x\geqq A$. Therefore it is enough to show that (1.1) determines uniquely $\phi(x)$ for $x<A$. Repeating Szabłowski's argument we differentiate (1.1) obtaining $2\frac{d}{dx}[c(x)\,\phi(x)]=-m\,\phi(x)$ for $0\leqq x<A$ or $2\frac{d\psi}{dx}=-m\psi(x)/c(x)$, where $\psi(x)=c(x)\,\phi(x)$. Thus

$$\psi(x)=C\exp-\int_0^x\frac{m}{2c(t)}\,dt$$

for each $0\leqq x<A$ and

$$\phi(x)=C/c(x)\exp-\int_0^x\frac{m}{2c(t)}\,dt.$$

The constant C is determined uniquely by $\int_0^A x^{n-m-1}\,\phi(x^2)\,dx=1$ and thus the theorem is proved.

In the case, when $\alpha=2r$ is an even integer, the reasoning used in the proof of Theorem 1.2 leads to differential equation

$$\frac{d^r}{dx^r}\psi(x)=(-1)^r(r-1)!\left(B\left(r,\frac{m}{2}\right)c(x)\right)^{-1}\psi(x). \tag{1.7}$$

This equation can be solved effectively for some values of r and some functions $c(x)$, but it seems to be hard to deal with it in full generality.

Note however that in the special case $n=2$, $m=1$ and $\alpha=2r\geqq 2$ being an even integer, one has the following integral equation for $\phi(x)$

$$c(x)\phi(x)$$
$$=1/B(r,m/2)\sum_{k=0}^{r-1}\binom{r-1}{k}(-1)^k x^k m_{r-k-1}-1/B(r,m/2)\int_0^x (y-x)^{r-1}\phi(y)\,dy, \tag{1.8}$$

where $m_k=\int_0 x^k\phi(x)\,dx$ is absolute moment of order $2k+1$ of X_2, $0\leqq k\leqq r-1$. Thus under appropriate restrictions on the function $c(x)$ one can infer that $c(x)$ together with all moments of orders $2k+1$, $0\leqq k\leqq r-1$ determine the distribution of (X_1, X_2) uniquely (see Pogorzelski (1966) Chap. 1, p. 13 for situations in which the Eq. (1.8) has a unique solution).

Fortunately at least in the particular case of uniform distribution on the sphere, it is possible to have uniqueness based on just a single moment.

Theorem 1.3. *If* $\mathbf{X}$ *has spherically invariant distribution such that for some real* $\alpha>0$ $E|X_1|^\alpha<\infty$, *and for some* $A>0$

$$E(\|\mathbf{X}_1\|^\alpha|\mathbf{X}_2)=\begin{cases}(A^2-\|\mathbf{X}_2\|^2)^{\alpha/2} & \text{if } \|\mathbf{X}_2\|<A,\\ 0 & \text{if } \|\mathbf{X}_2\|>A.\end{cases}$$

and moreover $m/2-n/4\notin N$ *and*

$$E\|\mathbf{X}_1\|^\alpha=A^\alpha\Gamma(n/2)\Gamma((m+\alpha)/2)/(\Gamma((n+\alpha)/2)\Gamma(m/2)),$$

then $\mathbf{X}$ *has uniform distribution on the sphere of radius* A.

We don't know, whether the condition $m/2-n/4\notin N$ can be omitted in general, but one can show that it is irrelevant at least if $\alpha/2$ is an odd integer.

Proof of Theorem 1.3. First observe that as in the proof of Theorem 1.1. $P(\mathbf{X}=0)=0$. Indeed, if $P(\mathbf{X}=0)\neq 0$, then $P(E(\|\mathbf{X}_1\|^\alpha|\mathbf{X}_2)=0)>0$ and since $E(\|\mathbf{X}_1\|^\alpha|\mathbf{X}_2=0)\neq 0$, this is not possible.

Therefore by Lemma 1.4 it is enough to show that the integral Eq. (1.1) determines uniquely the distribution of $\mathbf{X}$. First, observe that as in the proof of Theorem 1.2 $\phi(x)=0$ for almost all $x\geqq A$. Taking this into account and substituting $\psi(x)=\phi(A(1-x))$, where $0<x<1$ we can rewrite (1.1) in the form

$$x^\beta\psi(x)=K\int_0^x (x-y)^{\beta-1}\psi(y)\,dy, \qquad 0<x<1 \tag{1.9}$$

where $\beta=\alpha/2$, $K=1/B(m/2,\beta)$.

Convoluting both sides of Eq. (1.9) with the function $g(x)=x^r$, $x>0$ and evaluating at the point $x=1$ we obtain

$$\int_0^1 (1-x)^r x^\beta \psi(x)\,dx$$
$$=\Gamma(m/2+\beta)\,\Gamma(r+1)\,(\Gamma(m/2)\,\Gamma(\beta+r+1))^{-1}\int_0^1 (1-x)^{r+\beta}\psi(x)\,dx. \qquad (1.10)$$

Note that $\int_0^1 (1-x)^{(n-m)/2-1}(1-x)^a x^\beta \psi(x)\,dx=2E(\|\mathbf{X}_2\|^{2a}(A-\|\mathbf{X}_2\|^2)^\beta)$. Therefore (1.10) applied to $r=k+(n-m)/2-1$ gives

$$E(\|\mathbf{X}_2\|^{2k}(A^2-\|\mathbf{X}_2\|^2)^\beta)$$
$$=\Gamma(m/2+\beta)\,\Gamma(r+1)\,(\Gamma(m/2)\,\Gamma(\beta+r+1))^{-1}\,E(A^2-\|\mathbf{X}_2\|^2)^{k+\beta}. \qquad (1.11)$$

Using the binomial formula the left hand side of (1.11) can be written in the form

$$\sum_{j=0}^{k}\binom{k}{j}A^{2k-2j}(-1)^j E(A^2-\|\mathbf{X}_2\|^2)^{j+\beta}$$

giving the following expression for the numbers $\mathbf{m}_j=E(A^2-\|X_2\|^2)^{j+\beta}$, $j=0,1,\ldots$:

$$[\Gamma(m/2+\beta)\,\Gamma(r+1)\,(\Gamma(m/2)\,\Gamma(\beta+r+1))^{-1}-(-1)^j]\,\mathbf{m}_k$$
$$=\sum_{j=0}^{k-1}\binom{k}{j}A^{2k-2j}(-1)^j\mathbf{m}_j,\qquad k=1,2,\ldots. \qquad (1.12)$$

It remains to note that from (1.12) we can determine $\mathbf{m}_k$, $k\geqq 1$ uniquelly, since the coefficient $\Gamma(m/2+\beta)\,\Gamma(r+1)\,(\Gamma(m/2)\,\Gamma(\beta+r+1))^{-1}-(-1)^k$ is clearly non-zero if k is an odd integer and it is also non-zero for even k because we have assumed that $m/2-n/4\notin N$.

The recurrence relation obtained determines uniquely all the moments $E(A^2-\|\mathbf{X}_2\|^2)^{k+\beta}$ for $k=1,2,\ldots$ provided that $E(A^2-\|\mathbf{X}_2\|)^\beta$ $=E(E(\|\mathbf{X}_1\|^\alpha\,|\,\mathbf{X}_2))=E\,\|\mathbf{X}_1\|^\alpha$ is given.

Hence, see Shohat, Tamarkin (1943), p. 109, the distribution of $A^2-\|\mathbf{X}_2\|^2$ and the distribution of $\mathbf{X}$, is determined uniquely. The proof is completed.

The next result can be viewed as a characterization of the normal distribution among spherically invariant distributions by a kind of "linearity of regression" assumption expressed by (1.13) below.

Theorem 1.5. *Suppose that* $\mathbf{X}$ *is spherically invariant random vector with* $n\geqq 3$ *and let* $m_1, m_2\geqq 1$ *be such that* $m_1+m_2<n$. *Define* $\mathbf{X}_1=(X_1,\ldots,X_{m_1})$, $\mathbf{X}_2$

$=(X_{m_1+1}, \ldots, X_{m_1+m_2})$, $\mathbf{X}_3=(X_{m_1+m_2+1}, \ldots, X_n)$, *and suppose that for some real* $\alpha>0$ *we have* $E|X_1|^\alpha<\infty$ *and for some* $0<\beta<1$

$$E(\|\mathbf{X}_1\|^\alpha|\mathbf{X}_2, \mathbf{X}_3)=\beta E(\|\mathbf{X}_1\|^\alpha|\mathbf{X}_2)+(1-\beta)E(\|\mathbf{X}_1\|^\alpha|\mathbf{X}_3). \tag{1.13}$$

If $\sigma(\mathbf{X}_2)\cap\sigma(\mathbf{X}_3)$ *is trivial, then* $\mathbf{X}$ *is the Gaussian vector.*

Proof. By Theorem 1.1 it suffices to show that $E\,\|(\mathbf{X}_1\|^\alpha|\mathbf{X}_2)=E\,\|\mathbf{X}_1\|^\alpha$ a.e. Denote $E^Y(\cdot)=E(\cdot\,|\,Y)$. An "Alterniende Verfahren", see Rota (1962), says that $(E^{\mathbf{X}_2}E^{\mathbf{X}_3})^k\|\mathbf{X}_1\|^\alpha\to E(\|\mathbf{X}_1\|^\alpha|\sigma(\mathbf{X}_3)\cap\sigma(\mathbf{X}_2))$ almost everywhere as $k\to\infty$ (and hence in L_1, too). Since $\sigma(\mathbf{X}_3)\cap\sigma(\mathbf{X}_2)$ consists of sets of measure 0 or 1 only, the theorem will be proved by passing to the limit as $k\to\infty$, once we show the following lemma.

Lemma 1.6. *If* $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ *are random vectors such that for some* $0<\beta<1$ (1.13) *holds, then for each integer* $k\geqq 0$

$$E^{\mathbf{X}_2}(\|\mathbf{X}_1\|^\alpha)=(E^{\mathbf{X}_2}E^{\mathbf{X}_3})^k E^{\mathbf{X}_2}\|\mathbf{X}_1\|^\alpha. \tag{1.14}$$

Proof. We shall proceed by induction with respect to k. There is nothing to prove for $k=0$ and assume that (1.14) holds for some $k\geqq 0$. Then

$$E^{\mathbf{X}_2}\|\mathbf{X}_1\|^\alpha=E^{\mathbf{X}_2}E^{\mathbf{X}_2,\mathbf{X}_3}\|\mathbf{X}_1\|^\alpha=\beta E^{\mathbf{X}_2}\|\mathbf{X}_1\|^\alpha+(1-\beta)E^{\mathbf{X}_2}E^{\mathbf{X}_3}\|\mathbf{X}_1\|^\alpha.$$

Since $1-\beta\neq 0$ this implies

$$E^{\mathbf{X}_2}\|\mathbf{X}_1\|^\alpha=E^{\mathbf{X}_2}E^{\mathbf{X}_3}\|\mathbf{X}_1\|^\alpha. \tag{1.15}$$

Repeating the same reasoning with the use of (1.13) and the fact, that $\beta\neq 0$ we obtain

$$E^{\mathbf{X}_2}E^{\mathbf{X}_3}\|\mathbf{X}_1\|^\alpha=E^{\mathbf{X}_2}E^{\mathbf{X}_3}E^{\mathbf{X}_2}\|\mathbf{X}_1\|^\alpha. \tag{1.16}$$

Substituting (1.16) and (1.15) into (1.14) proves that (1.14) holds for $k+1$ and the lemma is proved by induction.

Remark. The case of an infinite sequence $\mathbf{X}=(X_k)_{k\in N}$ such, that for each n the distribution of $(X_1, \ldots, X_n)$ is spherically invariant, deserves separate mention because proofs can be then simplified. In that case it is known, that $\mathbf{X}=R\mathbf{G}$, where $\mathbf{G}=(G_k)$ is a sequence of i.i.d. Gaussian random variables and $R\geqq 0$ is a random variable independent of $\mathbf{G}$ and $\sigma(X_n, X_{n+1}, \ldots)$ measurable for each $n\geqq 1$, see e.g., Cambanis et al. (1981). Therefore if we split $\mathbf{X}$ into $(\mathbf{X}_1, \mathbf{X}_2)$ with $\mathbf{X}_1=(X_1, \ldots, X_n)$, $\mathbf{X}_2=(X_{n+1}, X_{n+2}, \ldots)$, we can easily see, that $E(\|\mathbf{X}_1\|^\alpha|\mathbf{X}_2)=E(R^\alpha E(\|\mathbf{G}_1\|^\alpha|\mathbf{G}_2, R)|\mathbf{X}_2)=\text{const}\,R^\alpha$. In particular this proves immediately Theorem 1.1, since the conditional moment is non-random iff $R=\text{const}$ is non-random. Also Eq. (1.13) is satisfied in this conditionally Gaussian situation.

2. Exponential Moments

It was shown in Bryc (1985) that if random variables X_0, X_1 are α-integrable for some $\alpha>0$ and such that for some $0<|\rho|<1$

$$E(|X_i-\rho X_j|^\alpha \mid X_j)\leqq C \text{ a.e.} \quad i\neq j \; i,j=0,1 \tag{2.1}$$

then X_0, X_1 have finite moments of all orders. In general the finiteness of exponential moments doesn't follow from (2.1), see example 2.3 below. However, one can strengthen condition (2.1) enough to prove finiteness of exponential moments. The statement of the condition might look a little artificial but it applies nicely to some situations including spherically invariant vectors.

Theorem 2.1. *Let C, α, $\varepsilon\geqq 0$, K be fixed positive constants, $\varepsilon<\alpha$ and assume X_1 is an α-integrable random variable, such that for each value of $|\rho|<1$ there is an α-integrable random variable X_ρ satisfying the conditions*

(a) $E(|X_1-|\rho|X_\rho|^\alpha \mid X_\rho)\leqq C(1-|\rho|)^\varepsilon$ a.e.
(b) $E(|X_\rho-|\rho|X_1|^\alpha \mid X_1)\leqq C(1-|\rho|)^\varepsilon$ a.e.
(c) $|X_{-\rho}|+|X_\rho|\geqq 2|\rho|\,|X_1|$
(d) $P(|X_\rho|>t)\leqq KP(|X_1|>t)$ *for each* $t\geqq 1$.

Then for some $\lambda>0$ $E\{\exp(\lambda|X_1|^{\alpha/(\alpha-\varepsilon)})\}<\infty$.

The proof of the theorem is based on the following lemma.

Lemma 2.2. *Under the assumption of Theorem 2.1 the function $N(t)=P(|X_1|>t)$ satisfies for each $a\geqq 0$ and $t>a$ the inequality*

$$a^\alpha N(t)\leqq 2K\sigma N(t-a)+2\sigma(K+1)N(t), \tag{2.2}$$

where $\sigma=C(a/t)^\varepsilon$.

Proof. Fix $0\leqq a<t$ and let $\rho=1-a/t$. Clearly

$$\begin{aligned} N(t)\leqq\, & P(|X_1|>t,\,|X_\rho|>t)+P(|X_1|>t,\,|X_{-\rho}|>t)\\ & +P(|X_1|>t,\,|X_\rho|\leqq t,\,|X_{-\rho}|\leqq t). \end{aligned} \tag{2.3}$$

We shall consider each term separately starting from the last one.

$$\begin{aligned} & P(|X_1|>t,\,|X_\rho|\leqq t,\,|X_{-\rho}|\leqq t)\leqq P(|X_1-\rho X_\rho|>(1-\rho)\,t,\\ & |X_1-\rho X_{-\rho}|>(1-\rho)\,t,\,|X_\rho|\leqq t,\,|X_{-\rho}|\leqq t,\,|X_\rho|+|X_{-\rho}|>2\rho t)\\ & \quad \leqq P(|X_1-\rho X_\rho|>(1-\rho)\,t,\,\rho t<|X_\rho|\leqq t)\\ & \qquad +P(|X_1-\rho X_{-\rho}|>(1-\rho)\,t,\,\rho t<|X_{-\rho}|\leqq t). \end{aligned}$$

Thus by a conditional version of Čebyšev inequality and (a), (b)

$$P(|X_1|>t,\,|X_\rho|\leqq t,\,|X_{-\rho}|\leqq t)\leqq 2K\sigma(1-\rho)^{-\alpha}t^{-\alpha}N(\rho t). \tag{2.4}$$

The first two right hand side terms in (2.3) can be bounded as follows, see Bryc (1985)

$$P(|X_1|>t, |X_\rho|>t) \leqq P(|X_1-\rho X_\rho|>(1-\rho)t, |X_\rho|>t) + P(|X_\rho-\rho X_1|>t, |X_1|>t) \leqq \sigma(K+1)[(1-\rho)t]^{-\alpha} N(t) \tag{2.5'}$$

and similarly

$$P(|X_1|>t, |X_{-\rho}|>t) \leqq \sigma(K+1)[(1-\rho)t]^{-\alpha} N(t). \tag{2.5''}$$

Inequality (2.2) follows now from (2.3), (2.4), (2.5) by our choice of ρ.

Proof of Theorem 2.1. Suppose first that $\varepsilon=0$, i.e. we have $\sigma=C$ in (2.2). Let a be such that $a^\alpha>2\sigma(K+1)$. Then (2.2) implies

$$N(t+a) \leqq qN(t) \tag{2.6}$$

where $q=\dfrac{2KC}{a^\alpha-2C(K+1)}$. By a choice of a large enough we can also ensure $q=e^{-\theta}$ for some $\theta>0$. Then (2.6) implies $N(na) \leqq e^{-\theta n} N(0)$, which in turn implies $N(t) \leqq \text{const}\, e^{-\frac{\theta}{a}t}$ for all $t \geqq 0$ and thus $E\exp\lambda|X_1|<\infty$ for some $\lambda>0$.

In the case when $\varepsilon \neq 0$, put $r=\alpha/\varepsilon$. Then (2.2) implies that for some constant A

$$N(t) \leqq A t^{-\varepsilon} a^{\varepsilon-\alpha} N(t-a), \qquad t \geqq a.$$

Choosing $a=(A+1)^{1/(\varepsilon(r-1))} t^{-1/(r-1)})$ for large enough t we obtain

$$N(t) \leqq \frac{A}{A+1} N(t-(A+1)^{1/(\varepsilon(r-1))} t^{-1/(r-1)})$$

which can be rewritten in the form

$$N(t) \leqq qN(t-Bt^{-1/(r-1)}) \qquad \text{for all } t \geqq T \tag{2.7}$$

where $0<q<1$ and T, $B>0$ are constants. Let $K=\dfrac{r}{r-1}B$. Then for any integer $n \geqq 1$,

$$(Kn)^{\frac{r-1}{r}} \leqq [K(n+1)]^{\frac{r-1}{r}} - B[K(n+1)]^{-1/r}. \tag{2.8}$$

Indeed (2.8) is equivalent to

$$(Kn)^{1-1/r}(Kn+K)^{1/r} \leqq Kn+K-B$$

and the last inequality is a consequence of elementary $x^{1/r}y^{1-1/r} \leqq \frac{1}{r}x+\left(1-\frac{1}{r}\right)y$ valid for any positive x, y and any exponent $0 \leqq \frac{1}{r} \leqq 1$. Since $N(t)$ is a decreasing function, it follows from (2.7) and (2.8) that $N([K(n+1)]^{(r-1)/r}) \leqq qN((Kn)^{(r-1)/r})$

for all n large enough. Therefore $N((Kn)^{\frac{r-1}{r}}) \leqq \text{const}\, q^n$ and hence $E(\exp \lambda |X_1|^{r/(r-1)}) < \infty$ for some $\lambda > 0$.

The following result follows immediately from Theorem 2.1.

Theorem 2.2. *Let* $(X_1, \ldots, X_n)$ *be a spherically symmetric* α*-integrable sequence,* $n \geqq 2$, $\alpha > 0$, *and for some* $1 \leqq m < n$ *denote* $\mathbf{X}_1 = (X_1, \ldots, X_m)$, $\mathbf{X}_2 = (X_{m+1}, \ldots, X_n)$. *If* $E(\|\mathbf{X}_1\|^\alpha | \mathbf{X}_2) \leqq \text{const}$ a.e. *then for some* $\lambda > 0$ $E \exp \lambda |X_1|^2 < \infty$.

Remark. Theorem 2.2 can be also obtained directly from Lemma 1.4. This approach requires estimating $\Gamma(n\alpha)$, $n \geqq 1$, which is well known in the case of integer α or, as was pointed out by J. Chen, using "integral inequalities" and Theorem 1.1. However the proof via Theorem 2.1 has a nice interpretation that for finiteness of exponential moments only values $\rho \to 1$ are essential. This seems to indicate an interesting difference from what was observed in Bryc, Szabłowski (1984) Theorem 2.1, where $\rho \to 0$ was used to determine the Gaussian distribution uniquely.

Proof. We shall reduce the problem to a pair of spherically symmetric variables. Indeed $E(|X_1|^\alpha | X_n) \leqq E(\|\mathbf{X}_1\|^\alpha | X_n) = E(E(\|\mathbf{X}_1\|^\alpha | \mathbf{X}_2) | X_n) \leqq \text{const}$. Since a pair (X_1, X_n) has spherically symmetric distribution, too, see Cambanis et al. (1981), without losing generality we may consider a pair of random variables only. Let θ be such that $|\rho| = \cos\theta$ and define for all $-1 < \rho < 1$

$$X_\rho = X_1 \cos\theta + X_n \sin\theta, \qquad \text{where sign}\,\theta = \text{sign}\,\rho.$$

Then the assumptions (c) and (d) of Theorem 2.1 are clearly satisfied. Also the assumptions (a) and (b) are satisfied with $\varepsilon = \alpha/2$. Indeed

$$E(|X_\rho - |\rho| X_1|^\alpha | X_1) = |\sin\theta|^\alpha E(|X_1|^\alpha | X_n) \leqq 2^{\alpha/2}(1 - |\rho|)^{\alpha/2} \text{ const}$$

thus (b) is satisfied.

Since $(X_1, X_\rho) \simeq (X_\rho, X_1)$ in distribution, also (a) is satisfied and by Theorem 2.1 $E \exp(\lambda |X_1|^2) < \infty$ for some $\lambda > 0$. The proof is completed.

Remark. In the case $\alpha = 2$ more detailed result was proved in the preliminary version of Szabłowski (1986a), see also Szabłowski (1987) Theorem 2.

We conclude with example, which shows that in the statement Theorem 2.1 one cannot limit the attention to a single value of ρ only and thus Bryc (1985) Theorem A cannot be improved to cover exponential moments.

Example 2.3. Consider random variables $(X_1, X_2) \simeq (X_2, X_1)$ with distribution concentrated on lines $y = \frac{1}{4}x$ and $y = 4x$, such that for $x \geqq 1$ the projection of the line $y = \frac{1}{4}x$ onto the x-axis has as probability density function $p(x) = Cx^{-\log_2 x}$, $x \geqq 1$, C being normalization constant. Then for $X_2 \geqq 1$

$$\begin{aligned} E(|X_1 - 1/4 X_2| \,|\, X_2 = x) &= (15/4)^2 x^2 p(4x)/(p(x) + p(4x)) \\ &\leqq 16(x^2 + 1/x^2)^{-1} \leqq 16, \; x \geqq 1. \end{aligned}$$

Since for values of $X_2 \leqq 1$ we have obviously $E(|X_1 - 1/4X_2| \mid X_2 = x) \leqq 16$, too, and by symmetry $E(|X_2 - 1/4X_1| X_1) \leqq 16$ a.e., this shows that the assumptions of Bryc (1985) Theorem A are satisfied. However, $E(\exp(\lambda|X_1|) = \infty$ for any choice of $\lambda > 0$.

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