Z. Wahrscheinlichkeitstheorie verw. Gebiete
46, 221–225 (1979)

Zeitschrift für
Wahrscheinlichkeitstheorie
und verwandte Gebiete

On the Conditional Expectation with Respect to a Sequence of Independent σ-fields

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Summary. In the paper we characterize those sequences of random variables which are conditional expectations of a p-integrable random variable with respect to a given sequence of independent σ-fields.

Let $(\Omega, \mathfrak{M}, P)$ be a probability space, and $(\mathfrak{N}_i)$ a sequence of independent σ-subfields of $\mathfrak{M}$ (e.g. for each sequence of $A_i \in \mathfrak{N}_i$ $i=1, \ldots, n$ there holds

$$P\left(\bigcap_{i=1}^{n} A_i\right) = \prod_{i=1}^{n} P(A_i)).$$

If $\mathfrak{N} \subset \mathfrak{M}$ and $1 \leqq p \leqq \infty$ we shall denote by $L_p(\mathfrak{N})$ (resp. L_p if $\mathfrak{N} = \mathfrak{M}$) the Banach space of all random variables X which are $\mathfrak{N}$-measurable and such that $\|X\|_p = (E|X|^p)^{1/p} < \infty$ if $p < \infty$, and

$$\|X\|_p = \operatorname*{supess}_{\omega \in \Omega} |X(\omega)| \quad < \infty \quad \text{if } p = \infty.$$

The closed linear subspace of $L_p(\mathfrak{N})$ of those X such that $EX = 0$ will be denoted by $L_p^0(\mathfrak{N})$ (resp. L_p^0 if $\mathfrak{N} = \mathfrak{M}$).

Theorem 1. a) *If X is a random variable such that $EX = 0$ and $E|X| \ln^+ |X| < \infty$ then the series $\sum_{i=1}^{\infty} E(X|\mathfrak{N}_i)$ is convergent in L_1 and almost surely.*

b) *If $X \in L_p^0$, and $1 < p < \infty$ then the series $\sum_{i=1}^{\infty} E(X|\mathfrak{N}_i)$ is convergent in L_p.*

Moreover there exists a constant C_p depending only on p such that

$$\left\| \sum_{i=1}^{\infty} E(X|\mathfrak{N}_i) \right\|_p \leqq C_p \|X\|_p.$$

0044-3719/79/0046/0221/$01.00

Proof. Let S be a linear operator defined by $S(X)=\sum_{i=1}^{\infty} E(X|\mathfrak{N}_i)$. Since $(\mathfrak{N}_i)_{i\in N}$ are independent σ-fields S is an orthogonal projection in L_2^0 and

$$\|S(X)\|_2^2=\sum_{i=1}^{\infty}\|E(X|\mathfrak{N}_i)\|_2^2\leqq\|X\|_2^2 \quad \text{for } X\in L_2^0.$$

By Kolmogorov theorem the series $\sum_{i=1}^{\infty} E(X|\mathfrak{N}_i)$ is convergent almost surely.

Let us define Banach spaces $L^0\ln$, $L^0\exp$ as follow: $L^0\ln$ (resp. $L^0\exp$) consists of all random variables such that $EX=0$ and $E|X|\ln^+|X|<\infty$ (resp. $E\exp\lambda|X|<\infty$ for some $\lambda>0$). The norms are defined as usually in Orlicz spaces. The bilinear form $\langle X, Y\rangle=EXY$ establishes an isomorphism between $L^0\exp$ and the dual space of $L^0\ln$ (cf. [5]). The following Lemma and the closed graph theorem imply that S is a continuous linear operator from L_∞^0 into $L^0\exp$.

Lemma 1. *If $(X_i)_{i\in N}$ is a sequence of independent, uniformly bounded random variables and the series $\sum_{i=1}^{\infty} X_i$ is convergent almost surely then $E\exp\lambda\left|\sum_{i=1}^{\infty} X_i\right|<\infty$ for each λ.*

The proof of Lemma follows from Hoffmann-Jørgensen's inequalities (cf. [2]) and was explicity given by Krakowiak, [4].

Since S is a selfadjoint operator S is a continuous linear operator from $L^0\ln$ into L^1. This proves that the series $\sum_{i=1}^{\infty} E(X|\mathfrak{N}_i)$ is convergent in L_1 and hence almost surely. To prove the second part of Theorem 1 let us consider an operator H defined by

$$H(X)=\sup_i |E(X|\mathfrak{N}_i)|.$$

H is a subadditive and positively homogeneous operator. Moreover

$$\|H(X)\|_\infty\leqq\|X\|_\infty$$

and

$$\|H(X)\|_2\leqq\|H(X-EX)\|_2+|E(X)|\leqq\|S(X-E(X)\|_2+\|X\|_2\leqq 2\|X\|_2.$$

Therefore by Marcinkiewicz interpolation theorem, [6], for $2\leqq p\leqq\infty$ there exists a constant C_p depending only on p such that

$$\|H(X)\|_p\leqq C_p\|X\|_p \quad \text{for } X\in L_p.$$

Lemma 2. *If $(X_i)_{i\in N}$ is a sequence of independent random variables and $p<\infty$ then*

$$\left\|\sum_{i=1}^{\infty} X_i\right\|_p\leqq K_p\left(\|\sup_i|X_i|\|_p+\left\|\sum_{i=1}^{\infty} X_i\right\|_2\right)$$

where K_p is a constant depending only on p.

The proof of Lemma 2 follows directly from Theorem 3.1 of Hoffmann-Jørgensen [2].

Now Lemma 2 and the preceding inequality give

$$\left\| \sum_{i=1}^{\infty} E(X|\mathfrak{N}_i) \right\|_p \leqq K_p(\|H(X)\|_p + \|S(X)\|_2) \leqq K_p(C_p\|X\|_p + \|X\|_p) =$$

$K_p(C_p+1)\|X\|_p$ if $2 \leqq p < \infty$, and $X \in L_p^0$. Therefore S is a continuous operator from L_p^0 into L_p^0 for $2 \leqq p < \infty$. Since S is a selfadjoint operator by duality arguments S is continuous also for $1 < p \leqq 2$. This completes the proof.

Remark 1. Theorem 1 generalizes Basterfield's, [1], result who proved that if $X \in L\ln$ then $E(X|\mathfrak{N}_i)$ is convergent almost surely to EX.

Corollary 1. *Let $1 < p < \infty$, and let $(X_i)_{i\in N}$ be a sequence of random variables such that $X_i \in L_p^0(\mathfrak{N}_i)$ $i = 1, 2, \ldots$. Then there exists $X \in L_p^0$ such that $X_i = E(X|\mathfrak{N}_i)$ $i = 1, 2, \ldots$ if and only if the series $\sum_{i=1}^{\infty} X_i$ is convergent in L_p or equivalently $E\left(\sum_{i=1}^{\infty} X_i^2\right)^{p/2} < \infty$.*

Proof. By Theorem 1 the condition is necessary. On the other hand side if the series $\sum_{i=1}^{\infty} X_i$ is convergent in L_p then putting X to be equal to the sum of the series we obtain that X has the desired property. To end the proof let us observe that by Marcinkiewicz theorem [6] the series $\sum_{i=1}^{\infty} X_i$ is convergent in L_p, $1 \leqq p < \infty$ if and only $E\left(\sum_{i=1}^{\infty} X_i^2\right)^{p/2} < \infty$.

Theorem 2. *Let $(X_i)_{i\in N}$ be a sequence of random variables such that $X_i \in L_1^0(\mathfrak{N}_i)$ $i = 1, 2, \ldots$. Then $\lim_{i\to\infty} \|X_i\|_1 = 0$ is a necessary and sufficient condition for the existence of X in L_1^0 such that $X_i = E(X|\mathfrak{N}_i)$ $i = 1, 2, \ldots$.*

Proof. Let $(T_i)_{i\in N}$ be a sequence of operators in L_1^0 defined by $T_i(X) = E(X|\mathfrak{N}_i)$ $i = 1, 2, \ldots$. Then $\|T_i\| \leqq 1$ and for each $X \in L_1^0(\sigma(\mathfrak{N}_1 \cup \cdots \cup \mathfrak{N}_n))$ $T_i X = 0$ for $i > n$. Since $\bigcup_{i=1}^{\infty} L_1^0\left(\sigma\left(\bigcup_{i=1}^{n} \mathfrak{N}_i\right)\right)$ is a dense subset in $L_1^0\left(\sigma\left(\bigcup_{i=1}^{\infty} \mathfrak{N}_i\right)\right)$ we obtain that $\lim_{i\to\infty} T_i X = 0$ for each $X \in L_1^0\left(\sigma\left(\bigcup_{i=1}^{\infty} \mathfrak{N}_i\right)\right)$. If $X \in L_1^0$, then $T_i X = T_i Y$ where $Y = E\left(X|\sigma\left(\bigcup_{i=1}^{\infty} \mathfrak{N}_i\right)\right)$ and thus $\lim_{i\to\infty} T_i X = 0$. This proves the necessity of the condition.

Let us denote by $c_0(L_1^0(\mathfrak{N}_i)_{i\in N})$ the Banach space of all sequences $(X_i)_{i\in N}$ of random variables such that $X_i \in L_1^0(\mathfrak{N}_i)$ $i = 1, 2, \ldots$ and such that $\lim_{i\to\infty} \|X_i\|_1 = 0$. The norm in the space is defined by

$$\|(X_i)_{i\in N}\|_{1,\infty} = \sup_i \|X\|_1.$$

The dual of this space is isomorphic with the space $l_1(L^0_\infty(\mathfrak{N}_i)_{i\in N})$ – the space of all sequences $(X_i)_{i\in N}$ of random variables such that $\|(X_i)_{i\in N}\|_{\infty,1} = \sum_{i=1}^{\infty} \|X_i\|_\infty < \infty$ and such that $X_i \in L^0_\infty(\mathfrak{N}_i)$ $i=1,2,\ldots$. The isomorphism is established by the bilinear form $\langle (X_i)_{i\in N}, (Y_i)_{i\in N}\rangle = \sum_{i=1}^{\infty} EX_i Y_i$.

By the first part of this proof the operator T defined by $T(X) = (E(X|\mathfrak{N}_i))_{i\in N}$ is a continuous linear operator from L^0_1 into $c_0(L^0_1(\mathfrak{N}_i)_{i\in N})$.

To end the proof we have to show that the operator T is "onto". By Banach theorem T is "onto" if the adjoint operator T^* is an isomorphic embedding, e.g. there exists a constant C such that

$$C\|T^*((X_i)_{i\in N})\|_\infty \geqq \|(X_i)_{i\in N}\|_{\infty,1}.$$

But T^* is given by $T^*((X_i)_{i\in N}) = \sum_{i=1}^{\infty} X_i$ and the existence of $C(C=1)$ follows from the Lemma

Lemma 3. *If $(X_i)_{i\in N}$ is a sequence of independent random variables with $EX_i=0$ $i=1,\ldots$ then*

$$\left\|\sum_{i=1}^{\infty} X_i\right\|_\infty = \sum_{i=1}^{\infty} \|X_i\|_\infty.$$

The proof is simple and is omitted.

Remark. 2. Theorem 2 shows that Theorem 1 may not be extended on the case $p=1$. It proves even that there exists $X\in L^0_1$ such that the sequence $(E(X|\mathfrak{N}_i))_{i\in N}$ is not convergent almost surely. It was observed in [1].

Theorem 3. *Let $(X_i)_{i\in N}$ be a sequence of random variables such that $X_i\in L^0_\infty(\mathfrak{N}_i)$ $i=1,2,\ldots$. A necessary and sufficient condition for the existence of $X\in L^0_\infty$ such that $X_i = E(X|\mathfrak{N}_i)$ $i=1,2,\ldots$ is that*

$$\sup_i \|X_i\|_\infty < \infty \quad \textit{and} \quad \sum_{i=1}^{\infty} \|X_i\|_2^2 < \infty.$$

Proof. Let us consider the Banach space W of all sequences $(X_i)_{i\in N}$ of random variables such that $X_i\in L^0_\infty(\mathfrak{N}_i)$ $i=1,2,\ldots$, and such that

$$\|(X_i)_{i\in N}\|_W = \max\left\{\sup_i \|X_i\|_\infty, \left(\sum_{i=1}^{\infty} \|X_i\|_2^2\right)^{1/2}\right\} < \infty.$$

The bilinear form $\langle (X_i)_{i\in N}, (Y_i)_{i\in N}\rangle = \sum_{i=1}^{\infty} EX_i Y_i$ establishes an isomorphism between W and the dual space of V – the space of all sequences $(X_i)_{i\in N}$ of random variables which can be written as $(X_i)_{i\in N} = (Y_i + Z_i)_{i\in N}$ where $Y_i\in L^0_1(\mathfrak{N}_i)$, $Z_i\in L^0_2(\mathfrak{N}_i)$ $i=1,2,\ldots$ and

$$\sum_{i=1}^{\infty} \|Y\|_1 + \left(\sum_{i=1}^{\infty} \|Z_i\|_2^2\right)^{1/2} < \infty,$$

the norm $\|(X_i)_{i\in N}\|_V$ is the infimum of the sum over all such representation of $(X_i)_{i\in N}$.

Now Theorem 3 may be reformulated in a way that the operator T (defined in the proof of Theorem 2) is a continuous linear operator from L^0_∞ onto W. Let T' be an operator defined by $T'((X_i)_{i\in N})=\sum_{i=1}^{\infty} X_i$. Then the adjoint operator of T' is the operator T, and therefore by Banach theorem to end the proof it is enough to show that T' is an isomorphic embedding of V into L^0_1, that is there exists a constant C such that

$$C^{-1}\|(X_i)_{i\in N}\|_V \leqq \|T'((X_i)_{i\in N}\|_1 \leqq C\|(X_i)_{i\in N}\|_V.$$

If $(X_i)_{i\in N}=(Y_i+Z_i)_{i\in N}$, is a representation as before then

$$\begin{aligned}\|T^*(X_i)\|_1 &= \left\|\sum_{i=1}^{\infty}(Y_i+Z_i)\right\|_1 \leqq \left\|\sum_{i=1}^{\infty} Y_i\right\|_1 + \left\|\sum_{i=1}^{\infty} Z_i\right\|_2\\ &\leqq \sum_{i=1}^{\infty}\|Y_i\|_1+\left(\sum_{i=1}^{\infty}\|Z_i\|_2^2\right)^{1/2}.\end{aligned}$$

Thus

$$\|T'((X_i)_{i\in N})\|_1 \leqq \|(X_i)_{i\in N}\|_V.$$

The other side of the inequality is obtained from the

Lemma 4. *There exists a constant C such that for each sequence $(X_i)_{i\in N}$ of independent random variables with $EX_i=0$ there holds*

$$\begin{aligned}C\left\|\sum_{i=1}^{\infty} X_i\right\|_1 &\geqq \sum_{i=1}^{\infty}\|X_i'\|_1+\left\|\sum_{i=1}^{\infty} X_i''\right\|_2 \geqq \frac{1}{2}\left(\sum_{i=1}^{\infty}\|X_i'-EX_i'\|_1\right.\\ &\left.+\left(\sum_{i=1}^{\infty}\|X_i''-EX_i''\|_2^2\right)^{1/2}\right), \quad \text{where } X_i'=X_i I_{|X_i|>1},\\ X_i''&=X_i I_{|X_i|\leq 1} \qquad i=1,2,\ldots.\end{aligned}$$

The proof of this Lemma is contained implicitly in the proof of Theorem 3.6 of Jain, Marcus [3].

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Received December 13, 1977