Meixner matrix ensembles

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[1] Based on joint work with Gerard Letac

Outline of talk

- Random matrices
- Meixner laws
- System of PDEs
- 2×2 Meixner ensembles
- Conclusions

$\mathbf{X} = [X_{i,j}]$ an $m \times n$ matrix with random real complex or quaternion entries.

- physics (Wigner's semicircle law, enumeration of manifolds) $\boxed{> 1950}$
- statistics (distribution of eigenvalues of a sample covariance matrix) $\boxed{\leq 1928}$
- wireless communication (signal+noise+several antenas) $\boxed{=1997}$
- population genetics (500 000 dimensional observations of 1000 individuals) $\boxed{=2006}$

Notation

- Random matrix: r.v. $\mathbf{X}$ with values in the space $\mathbb{H}_{n,1}$ of all symmetric matrices, $\mathbb{H}_{n,2}$ of Hermitian complex matrices; $\mathbb{H}_{n,4}$ of Hermitian-quaternionic matrices
- For $\beta = 1, 2, 4$, random matrix $\mathbf{X}$ is "rotation invariant" if $\mathbf{X} \sim U\mathbf{X}U^*$ for all U in $\mathbb{O}(n)$, $\mathbb{U}(n)$, or $\mathbb{S}p(n)$.

Definition

A random matrix $\mathbf{X}$ is a Meixner ensemble if is rotation invariant and there exist $A, B, C \in \mathbb{R}$ such that for independent $\mathbf{X} \sim \mathbf{Y}$

$$\mathbf{E}((\mathbf{X}-\mathbf{Y})^2 | \mathbf{X}+\mathbf{Y}) = A(\mathbf{X}+\mathbf{Y})^2 + B(\mathbf{X}+\mathbf{Y}) + C\mathbf{I}_n \,. \tag{1}$$

First examples

- GOE/GUE/GSE is Meixner: $\mathbf{X}-\mathbf{Y}$ is independent of $\mathbf{S}=\mathbf{X}+\mathbf{Y}$ so $\mathbf{E}((\mathbf{X}-\mathbf{Y})^2|\mathbf{S})=C\mathbf{I}$.
- Wishart matrices are not Meixner:

$$\mathbf{E}((\mathbf{X}-\mathbf{Y})^2|\mathbf{S})=A_n\mathbf{S}^2+B_n\mathbf{S}\,\mathrm{tr}\,\mathbf{S}, \tag{2}$$

  [Letac-Massam-98]
- Trivial Meixner ensembles $\mathbf{X}=X\mathbf{I}_n$, where X is a real r.v., are described on next slide.

Trivial Meixner ensembles

Let X, Y have the same law, $E(X) = 0$, $E(X^2) = 1$, $S = X + Y$,

$$\text{Assume } \mathrm{Var}(X|S) = C(a,b)(1 + aS + bS^2)$$

	X, Y independent [Laha Lukacs (1960)]	X, Y free [Bożejko-B (2006)]
$b = -1/4$	Bernoulli	Bernoulli
$b < 0$	binomial	free binomial (McKay)
$a \neq 0$, $b = 0$	Poisson	Marchenko-Pastur
$a^2 > 4b$, $b > 0$	negative binomial	(no name)
$a^2 = 4b = 0$	Gaussian	Wigner's semicircle
$a^2 = 4b > 0$	gamma	(no name)
$a^2 < 4b$	hyperbolic secant	(no name)
Converse:	Yes	?

Anshelevich's question

In his 2006 talk at MIT, M. Anshelevich raised the question of defining Meixner distributions on matrices, and in particular asked for the matrix version of Laha-Lukacs (1960) result.

Question (Anshelevich, 2006)

What are the non-trivial laws on symmetric $n \times n$ matrices with the property that if $\mathbf{X}, \mathbf{Y}$ are independent, rotation invariant with the same law and $\mathbf{S} = \mathbf{X} + \mathbf{Y}$, then $\mathbf{E}(\mathbf{X}^2|\mathbf{S})$ is a real quadratic polynomial in $\mathbf{S}$, i.e., there are real constants A, B, C such that

$$\mathbf{E}((\mathbf{X} - \mathbf{Y})^2|\mathbf{S}) = A\mathbf{S}^2 + B\mathbf{S} + C\mathbf{I}_n \,. \tag{3}$$

Summary End now

Bernoulli ensemble

Denote by $\mathbf{P}_m$ the orthogonal projection onto the random and uniformly distributed m-dimensional subspace of $\mathbb{R}^n$, $\mathbb{C}^n$ or $\mathbb{H}^n$. Let

$$\mathbf{X} = \begin{cases} \mathbf{0} & \text{with probability } q_0 = 1 - (q_1 + \cdots + q_n)\,, \\ \mathbf{P}_1 & \text{with probability } q_1\,, \\ \vdots & \\ \mathbf{I}_n & \text{with probability } q_n\,. \end{cases} \tag{4}$$

Proposition

A Bernoulli ensemble is a Meixner ensemble:

$$\mathbf{E}((\mathbf{X}-\mathbf{Y})^2|\mathbf{S}) = A\mathbf{S}^2 + B\mathbf{S} + C\mathbf{I}_n$$

with $A = -1$, $B = 2$, $C = 0$.

For any pair of projections, $(P-Q)^2 = 2(P+Q) - (P+Q)^2$.

$$\mathbf{E}((\mathbf{X}-\mathbf{Y})^2|\mathbf{S}) = A\mathbf{S}^2 + B\mathbf{S} + C\mathbf{I}_n\,.$$

Proposition

Suppose that a law $\mathbf{X}$ *is a Meixner ensemble with parameters* $A=-1$, $B=2$, $C=0$, *i.e.,*

$$\mathbf{E}((\mathbf{X}-\mathbf{Y})^2|\mathbf{S}) = -\mathbf{S}^2 + 2\mathbf{S},$$

and that the first four moments are finite. Then $\mathbf{X}$ *is a Bernoulli ensemble.*

Plan of proof.

$\operatorname{tr}\mathbf{E}(\mathbf{X}^4) = \operatorname{tr}\mathbf{E}(\mathbf{X}^3) = \operatorname{tr}\mathbf{E}(\mathbf{X}^2)$ so $\mathbf{E}\operatorname{tr}(\mathbf{X}^2(\mathbf{I}-\mathbf{X})^2) = 0.$ □

▸ Summary ▸▸ End now

Binomial ensemble

Fix integer N and non-negative numbers $q_1, \dots, q_n$ with $q_1 + \dots + q_n \leq 1$. Let $\mathbf{X}_1, \dots, \mathbf{X}_N$ be independent random matrices with the same Bernoulli distribution (4).

Definition

The binomial ensemble $\mathrm{Bin}(N, q_1, \dots, q_n)$ is the law of $\mathbf{X} = \sum_{j=1}^N \mathbf{X}_j$.

Proposition

A binomial ensemble with parameter N is a Meixner ensemble with $A = -1/(2N-1)$, $B = 2N/(2N-1)$, $C = 0$.

▸ Summary ▸▸ End now

Method of proof

$$\mathbf{E}((\mathbf{X}-\mathbf{Y})^2|\mathbf{S}) = A\mathbf{S}^2 + B\mathbf{S} + C\mathbf{I}_n$$

is equivalent to

$$\mathbf{E}\left((\mathbf{X}-\mathbf{Y})^2 e^{\langle\theta|\mathbf{S}\rangle}\right) = \mathbf{E}\left((A\mathbf{S}^2 + B\mathbf{S} + C\mathbf{I})e^{\langle\theta|\mathbf{S}\rangle}\right)$$

This in turn is equivalent to a system of n PDEs which can be written in matrix form as

$$2(1-A)\Psi(k''(\theta))(\mathbf{I}_n) = 4A(k'(\theta))^2 + 2Bk'(\theta) + C\mathbf{I}_n \qquad (5)$$

for the log-Laplace transform $k(\theta) = \ln \mathbf{E} e^{\langle\theta|\mathbf{S}\rangle}$.

Summary End now

Proposition

If $\mathbf{X}$ *is a* 2×2 *matrix with finite exponential moments and such that*

$$\mathbf{E}((\mathbf{X}-\mathbf{Y})^2|\mathbf{S}) = A\mathbf{S}^2 + B\mathbf{S} + C\mathbf{I}_n$$

with $A<0$, $C=0$, *then there exists* $N \in \mathbb{N}$ *such that* $A=-1/(2N-1)$, $B=2N/(2N-1)$, $C=0$, *and* $\mathbf{X}=\mathbf{X}_1+\cdots+\mathbf{X}_N$ *is Binomial.*

Up to affine transformations, from the system of PDEs the Laplace transform is

$$\left(p\cosh(\alpha + \operatorname{tr}\theta) + (1-p)\mathcal{I}_{(\beta-1)/2}\left(\sqrt{\operatorname{tr}^2\theta - 4\det\theta}\right)\right)^{(A-1)/(2A)},$$

where $\mathcal{I}$ is a version of modified Bessel function, normalized so that $\mathcal{I}(0)=1$.

Example (2×2 Binomial ensemble)

$$\mathbf{P} = \frac{1}{2}\begin{bmatrix} 1+\cos T & \sin T \\ \sin T & 1-\cos T \end{bmatrix}$$

where e^{iT} is uniformly distributed on the unit circle. The binomial ensemble $\mathbf{X}_N = \mathbf{P}_1 + \dots \mathbf{X}_N$ has eigenvalues

$$\lambda_\pm = \frac{1}{2}(N \pm |e^{iT_1} + e^{iT_2} + \dots + e^{iT_N}|).$$

Random variable $\lambda_+ - \lambda_-$ has known distribution on $[0, N]$; $P(\lambda_+ - \lambda_- < 1) = \frac{1}{N+1}$, see [Spitzer:1964, page 104] .

Definition

The Poisson ensemble is the law of

$$\mathbf{X} = \sum_{k=0}^{N} \mathbf{X}_k,$$

where $\Pr(N = j) = e^{-\lambda}\lambda^j/j!$, $j = 0, 1, \dots$ and $\mathbf{X}_1, \mathbf{X}_2, \dots$ are independent Bernoulli matrices with the same $q_1, \dots, q_n$.

Proposition

The Poisson ensemble is a Meixner ensemble: $\mathbf{E}((\mathbf{X} - \mathbf{Y})^2 | \mathbf{S}) = \mathbf{S}$.
Among the 2×2 *matrices, these are the only such ensembles.*

▸ Summary ▸▸ End now

Definition

The negative binomial ensemble is the law of the random sum

$$\mathbf{X} = \sum_{k=0}^{M} \mathbf{X}_k \, , \tag{6}$$

where $\mathbf{X}_1, \mathbf{X}_2, \dots,$ are independent Bernoulli ensembles and

$$P(M = j) = \frac{\Gamma(r+j)}{\Gamma(r) j!} p^r q^j, \; q = 1 - p. \tag{7}$$

Proposition

The negative binomial ensemble is a Meixner ensemble, with $A = \frac{1}{2r+1}$, $B = \frac{2r}{2r+1}$, $C = 0$.
Among the 2×2 *matrices, these are the only such ensembles.*

Let $\mathbf{X}, \mathbf{Y} \in \mathbb{H}_{n,\beta}$ have the same law, $\mathbf{E}(\mathbf{X}) = 0$, $\mathbf{E}(\mathbf{X}^2) = \mathbf{I}$, $\mathbf{S} = \mathbf{X} + \mathbf{Y}$,

$$\text{Assume } \mathbf{E}((\mathbf{X} - \mathbf{Y})^2 | \mathbf{S}) = \frac{2}{1+2b}(1 + a\mathbf{S} + b\mathbf{S}^2)$$

	$n = 1$	$n = 2$	$n = 3$ (and more)	$n = \infty$
$b = -1/4$	Bern(q)	Bern(q_1, q_2)	Bern(q_1, q_2, q_3)	Bern(q_1)
$b < 0$	Bin(N, q)	Bin(N, q_1, q_2)	Bin(N, q_1, q_2, q_3)+?	fBin(T, q)
$a \neq 0,\ b = 0$	Poiss(λ)	Poiss(λ_1, λ_2)	Poiss$(\lambda_1, \lambda_2, \lambda_3)$+?	M-P(λ)
$a^2 > 4b > 0$	NB(r, q)	NB(r, q_1, q_2)	NB(r, q_1, q_3, q_n)+?	fNB(r, q)
$a^2 = 4b = 0$	Gauss	Gauss(c)	Gauss(c)+?	$\sqrt{4 - x^2}$
$a^2 = 4b > 0$	$\Gamma(r)$	$\Gamma(r, c)$	?	$f\Gamma(r)$
$a^2 < 4b$	HS(α)	HS(α), J(α)	?	fHS(α)
Converse?	Yes	Yes	yes...	? (some)

▸ Historical comments

Notation

- Consider random matrices $\mathbf{X} : \Omega \to \mathbb{H}_{n,\beta}$ with the Laplace transform

$$L(\theta) = \mathbf{E}(\exp\langle \theta | \mathbf{X} \rangle)$$

  - For $\theta \in \mathbb{H}_{n,\beta}$ and $i = 0, 1, \dots,$ we consider $\sigma_i(\theta)$ defined by

$$\det(\mathbf{I}_n + x\theta) = \sum_{i=0}^{\infty} \sigma_i(\theta) x^i.$$

  - $\sigma_0 = 1$, $\sigma_1(\theta) = \operatorname{tr} \theta$, $\sigma_2(\theta) = \frac{1}{2}(\operatorname{tr} \theta)^2 - \frac{1}{2} \operatorname{tr} (\theta^2)$, ... $\sigma_n(\theta) = \det \theta$.

Overview | End now

Theorem (PDEs for the Laplace transform)

Suppose $\mathbf{X} \in \mathbb{H}_{n,\beta}$ *has Laplace transform* $L(\theta) = \mathbb{E}(e^{\langle \theta | \mathbf{X} \rangle})$, *is invariant under rotations, and* $\mathbf{E}(\mathbf{X}) = \mathbf{0}$, $\mathbf{E}(\mathbf{X}^2) = \mathbf{I}$. *Suppose*

$$\mathbf{E}((\mathbf{X} - \mathbf{Y})^2 | \mathbf{S}) = C(\mathbf{I} + a\mathbf{S} + b\mathbf{S}^2)$$

Then $L(\theta)$ *can be expressed as a function of elementary symmetric functions of the eigenvalues of* θ: $\sigma_1(\theta), \dots, \sigma_n(\theta)$. *The following PDE holds after a substitution* $g(\sigma_1, \dots, \sigma_n)$ *for a function of* $L(\theta)$ *when* $\theta \in \Theta_0 \subset \Theta$. ...

Theorem (Generic case: $b \neq 0$, $n = 3$)

... $g(\sigma_1(\theta), \sigma_2(\theta), \sigma_3(\theta)) = e^{-a\,\mathrm{tr}\,\theta}\left(L(\theta)\right)^{-4b}$ *solves*

$$\begin{aligned}
\frac{\partial^2 g}{\partial \sigma_1^2} - \sigma_2 \frac{\partial^2 g}{\partial \sigma_2^2} - 2\sigma_3 \frac{\partial^2 g}{\partial \sigma_2 \partial \sigma_3} - \beta \frac{\partial g}{\partial \sigma_2} &= (a^2 - 4b) g \\
2\frac{\partial^2 g}{\partial \sigma_1 \partial \sigma_2} + \sigma_1 \frac{\partial^2 g}{\partial \sigma_2^2} - \sigma_3 \frac{\partial^2 g}{\partial \sigma_3^2} - \frac{\beta}{2} \frac{\partial g}{\partial \sigma_3} &= 0 \\
\frac{\partial^2 g}{\partial \sigma_2^2} + 2\frac{\partial^2 g}{\partial \sigma_1 \partial \sigma_3} + 2\sigma_1 \frac{\partial^2 g}{\partial \sigma_2 \partial \sigma_3} + \sigma_2 \frac{\partial^2 g}{\partial \sigma_3^2} &= 0
\end{aligned}$$

With conditions $g(0,0,0) = 1$, $\left.\frac{\partial g(\sigma_1, 0, 0)}{\partial \sigma_1}\right|_{\sigma_1 = 0} = -a$.

James (1955) shows that a very similar system of PDEs (with $\beta = 1$) has a unique solution analytic at 0, with $g(0) = 1$, and for $n = 3$ he gives the explicit series solution.

James system for $n = 3$ is the top three eqtns. Our system for $a^2 = 4b$ is the bottom three eqtns.

$$\begin{aligned}
\sigma_1 \frac{\partial^2 g}{\partial \sigma_1^2} + 2\sigma_2 \frac{\partial^2 g}{\partial \sigma_1 \partial \sigma_2} + 2\sigma_3 \frac{\partial^2 g}{\partial \sigma_1 \partial \sigma_3} + \sigma_3 \frac{\partial^2 g}{\partial \sigma_3^2} - \frac{3}{2} \frac{\partial g}{\partial \sigma_1} &= -\frac{1}{4} g \\
\frac{\partial^2 g}{\partial \sigma_1^2} - \sigma_2 \frac{\partial^2 g}{\partial \sigma_2^2} - 2\sigma_3 \frac{\partial^2 g}{\partial \sigma_2 \partial \sigma_3} - \frac{2}{2} \frac{\partial g}{\partial \sigma_2} &= 0g \\
2 \frac{\partial^2 g}{\partial \sigma_1 \partial \sigma_2} + \sigma_1 \frac{\partial^2 g}{\partial \sigma_2^2} - \sigma_3 \frac{\partial^2 g}{\partial \sigma_3^2} - \frac{1}{2} \frac{\partial g}{\partial \sigma_3} &= 0 \\
\frac{\partial^2 g}{\partial \sigma_2^2} + 2 \frac{\partial^2 g}{\partial \sigma_1 \partial \sigma_3} + 2\sigma_1 \frac{\partial^2 g}{\partial \sigma_2 \partial \sigma_3} + \sigma_2 \frac{\partial^2 g}{\partial \sigma_3^2} - \frac{0}{2} \frac{\partial g}{\partial \sigma_3} &= 0
\end{aligned}$$

Generic case: $b \neq 0$, $n = 2$

Domain: $U = \{(\sigma_1, \sigma_2) : \; 4\sigma_2 < \sigma_1^2\}$. PDEs:

$$\begin{aligned} \frac{\partial^2 g}{\partial \sigma_1^2} - \sigma_2 \frac{\partial^2 g}{\partial \sigma_2^2} - \frac{\beta}{2}\frac{\partial g}{\partial \sigma_2} &= (a^2 - 4b) g, \\ 2\frac{\partial^2 g}{\partial \sigma_1 \partial \sigma_2} + \sigma_1 \frac{\partial^2 g}{\partial \sigma_2^2} &= 0. \end{aligned}$$

$$g(0,0) = 1, \; \frac{\partial g(\sigma_1, 0)}{\partial \sigma_1}\Big|_{\sigma_1 = 0} = -a.$$

Solutions depend on $\kappa^2 = a^2 - 4b$

$$L(\theta) = e^{-\frac{a}{4b}\,\mathrm{tr}\,\theta}\left[g\left(\sigma_1(\theta), \sigma_2(\theta)\right)\right]^{-1/(4b)}$$

Overview | End now

Generic case: $b \neq 0$, $a^2 = 4b$

Proposition (Laplace transform for Gamma ensemble)

For $a^2 = 4b > 0$ all solutions are

$$g(\sigma_1, \sigma_2) = 1 - a\sigma_1 + C(\beta\,{\sigma_1}^2 + 4\sigma_2),$$

where C is an arbitrary constant.

Question

$$L(\theta) = e^{-\operatorname{tr}\theta/a} \frac{1}{(1 - a\operatorname{tr}\theta + C(\beta(\operatorname{tr}\theta)^2 + 4\det\theta))^{1/a^2}}$$

Is this a Laplace transform of a probability measure on $\mathbb{H}_{2,\beta}$?

Generic case: $b \neq 0$, $a^2 > 4b$

Proposition (Laplace transform for elliptic ensembles)

For $a^2 > 4b$ with $b \neq 0$ all solutions are

$$g(\sigma_1, \sigma_2) = (1 - C)\cosh|\kappa|\sigma_1 - \frac{a}{\kappa}\sinh\kappa\sigma_1 + C\mathcal{I}_\beta\left(\kappa\sqrt{\sigma_1^2 - 4\sigma_2}\right),$$

where C is an arbitrary constant, and $\mathcal{I}_\beta(z) = C_\beta \frac{I_{(\beta-1)/2}(z)}{z^{(\beta-1)/2}}$ (Modified Bessel I function)

Note: for $\theta = \begin{bmatrix} \theta_1 & 0 \\ 0 & \theta_2 \end{bmatrix}$, $\mathcal{I}_\beta\left(\kappa\sqrt{\sigma_1^2 - 4\sigma_2}\right) = \mathcal{I}_\beta\left(\kappa(\theta_1 - \theta_2)\right)$.

This is $\frac{\sinh(\kappa(\theta_1 - \theta_2))}{\theta_1 - \theta_2}$ when $\beta = 2$.

Generic case: $a^2 < 4b$

Proposition (Laplace transform for hyperbolic ensemble)

For $a^2 < 4b$ all solutions are

$$g(\sigma_1, \sigma_2) = (1 - C)\cos|\kappa|\sigma_1 - \frac{a}{\kappa}\sin\kappa\sigma_1 + C\mathcal{J}_\beta\left(\kappa\sqrt{\sigma_1^2 - 4\sigma_2}\right).$$

where C is an arbitrary constant,

$$\mathcal{J}_\beta(z) = C_\beta \frac{J_{(\beta-1)/2}(z)}{z^{(\beta-1)/2}}$$

(Bessel J function.)

Proposition

If $\mathbf{P}_1$ *is an random projection of* $\mathbb{H}_{n,\beta}$ *invariant by rotation with trace 1, then* $\mathbf{E}e^{\langle \theta | \mathbf{P}_1 \rangle} = L_n(\theta)$ *where*

$$L_n(\theta) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\left(\frac{n\beta}{2}\right)_k} \sum_{\nu_1+2\nu_2+3\nu_3+\dots=k} \frac{(-1)^{\nu_1+\nu_2+\nu_3+\dots} \left(\frac{n\beta}{2}\right)_{\nu_1+\nu_2+\nu_3+\dots}}{\nu_1!\nu_2!\nu_3!\dots} \sigma_1^{\nu_1}(\theta)\sigma \tag{8}$$

This can be used to write some solutions of the PDEs for the elliptic case.

Thank you

Six classical Meixner laws

Gaussian, Poisson, Gamma, Pascal (negative binomial), hyperbolic secant, *binomial*.

- [Meixner (1934)]: orthogonality measure of "Meixner orthogonal polynomials"
- [Tweedie (1946)], [Laha Lukacs (1960)]: laws with quadratic conditional variance $\mathrm{Var}(X|X+Y)$ for independent (i.i.d.) X, Y
- [Ismail May (1978)]: approximation operators
- [Morris (1982)]: exponential families with quadratic variance function

◂ Summary ▸▸ End now

Six free Meixner laws

Wigner's semicircle, Marchenko-Pastur, "free-Gamma", free-negative binomial, (un-titled), *free binomial (Kesten, McKay)*.
The absolutely continuous part of $\mu_{a,b}$ is

$$\frac{\sqrt{4(1+b)-(x-a)^2}}{2\pi(bx^2+ax+1)}$$

(may also have one or two atoms) .

- [Saitoh Yoshida (2001)]: orthogonality measure of orthogonal polynomials with "constant recurrence"
- [Anshelevich (2003)]: orthogonality measure of "free-Meixner orthogonal polynomials"
- [Bożejko Bryc (2006)]: laws with quadratic conditional variance $\mathrm{Var}(X|X+Y)$ for free X, Y
- [Bryc Ismail (arxiv 2005)], [Bryc (2009)]: Cauchy-kernel families with quadratic variance function

◂ Summary ▸▸ End now

Abstract

In this talk I will discuss random matrices that are matricial analogs of the well known binomial, Poisson, and negative binomial random variables. The defining property is that the conditional variance of $\mathbf{X}$ given the sum $\mathbf{S} = \mathbf{X} + \mathbf{X}'$ of two independent copies of $\mathbf{X}$ is a quadratic polynomial in $\mathbf{S}$; this property describes the family of six univariate laws on $\mathbb{R}$ that will be described in the talk, and we are interested in their matrix analogs. The talk is based on joint work with Gerard Letac.

◂ Summary ⏩ End now