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On Dominations between Measures of Dependence

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Suppose one has two measures of dependence between two or more families of random variables. One of the measures is said to "dominate" the other if the latter becomes arbitrarily small as the former becomes sufficiently small. A description is given of the entire pattern of dominations between arbitrary pairs of measures of dependence that are based on the usual norms of the bilinear form "covariance." Also, for a broader class of measures of dependence, some earlier "domination inequalities" are shown to be essentially sharp. © 1987 Academic Press, Inc.

1. INTRODUCTION

Throughout this paper, if $1 \leqslant p \leqslant \infty$ then p' denotes the conjugate exponent of p, i.e., $1 \leqslant p' \leqslant \infty$ and $1/p + 1/p' = 1$.

Suppose $(\Omega, \mathscr{M}, P)$ is a probability space. For any σ-field $\mathscr{F} \subset \mathscr{M}$ let $\mathscr{S}(\mathscr{F})$ denote the set of (equivalence classes of) complex-valued $\mathscr{F}$-measurable simple functions.

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Define the following measures of dependence between pairs of σ-fields $\mathscr{F}$ and $\mathscr{G} \subset \mathscr{U}$: For $0 \leqslant r, s \leqslant 1$,

$$\alpha_{r,s}(\mathscr{F}, \mathscr{G}) := \sup \frac{|P(A \cap B) - P(A)\, P(B)|}{[P(A)]^r\, [P(B)]^s}, \qquad A \in \mathscr{F},\ B \in \mathscr{G}. \quad (1.1)$$

For $1 \leqslant p, q \leqslant \infty$,

$$R_{p,q}(\mathscr{F}, \mathscr{G}) := \sup \frac{|Efg - Ef\,Eg|}{\|f\|_p\, \|g\|_q}, \qquad f \in \mathscr{S}(\mathscr{F}),\ g \in \mathscr{S}(\mathscr{G}). \quad (1.2)$$

Here and throughout this paper, 0/0 is interpreted to be 0. For $1 \leqslant p \leqslant \infty$, $\|f\|_p$ denotes the usual p-norm with respect to the given probability measure P. In (1.2) the restriction to simple functions is obviously unnecessarily strong; it is for convenience only. Note that $\alpha_{r,s}(\cdot, \cdot)$ is a "restricted" version of $R_{1/r, 1/s}(\cdot, \cdot)$, the restriction being to indicator functions.

If just real-valued functions are used in (1.2), then trivially the value of $R_{p,q}(\mathscr{F}, \mathscr{G})$ would decrease by at worst a factor of 4 (and by [18, Theorem 1.1] and simple arithmetic the value of $R_{2,2}(\mathscr{F}, \mathscr{G})$ would not change at all and is equal to the "maximal correlation" between $\mathscr{F}$ and $\mathscr{G}$). However, some applications of interpolation theory are "cleaner" in the complex case than in the real case (see, e.g., [10, Lemma 1; 3, Section 3], or Eq. (2.5) below), and therefore we shall use complex functions.

The measures of dependence $\alpha_{0,0}$, $\alpha_{1,0}$, $\alpha_{1,1}$, and $R_{2,2}$ are respectively the bases for the "strong mixing," "ϕ-mixing," "ψ-mixing," and "ρ-mixing" conditions for stochastic processes; see, e.g., Peligrad [12] for the definition of these mixing conditions.

If d_1 and d_2 are measures of dependence between pairs of σ-fields, we say that d_2 "dominates" d_1 if $d_1(\mathscr{F}, \mathscr{G})$ becomes arbitrarily small as $d_2(\mathscr{F}, \mathscr{G})$ becomes sufficiently small—that is, if there is a function $\Phi: [0, \infty] \to [0, \infty]$ with $\Phi(0) = 0$ and Φ continuous at 0, such that the inequality $d_1(\mathscr{F}, \mathscr{G}) \leqslant \Phi(d_2(\mathscr{F}, \mathscr{G}))$ holds for all pairs of σ-fields $\mathscr{F}$ and $\mathscr{G}$, in all probability spaces. Two measures of dependence are said to be "equivalent" if each one dominates the other.

For example, the measures $\alpha_{1/2, 1/2}$ and $R_{2,2}$ are equivalent, and hence $\alpha_{1/2, 1/2}$ is also a basis for the ρ-mixing condition. (See [2; 6; 3, Theorem 4.1(vi)].)

Many of the "moment inequalities" commonly used in central limit theory for mixing random variables (including, e.g., [9, Theorems 17.2.1, 17.2.2, and 17.2.3]) can be expressed in terms of dominations between measures of dependence $\alpha_{r,s}$ and $R_{p,q}$.

This note continues and complements [3]. Extending [3, Remark 4.1], Section 2 below gives a complete picture of the dominations between pairs of the measures of dependence $\alpha_{r,s}$ and $R_{p,q}$ in (1.1) and (1.2). In [3], in the endeavor to establish tight "domination inequalities" between various measures of dependence, the principal results were [3, Theorems 2.1, 2.2, 3.6, and 4.1(vi)]. All of these results turn out to be within a constant factor of being sharp, for any choice of parameters meeting the given specifications. (Such a constant factor may depend on the parameters.) This will be shown with a construction in Section 3 below. Finally, in Section 4 below, a short proof of [3, Theorem 2.2] will be given.

This note and [5] give disjoint pieces of [4]. The paper [5] studies measures of dependence similar to those in (1.2) but involving random variables taking their values in Hilbert spaces or Banach spaces; it exposes a very simple connection between those measures of dependence and the one on which the "absolute regularity" (weak Bernoulli) condition [17] is based, and it also extends [3, Theorem 4.2]. As a complement of [3, Section 4.4], in an unpublished section of [4], the "extreme point" method is used to establish the sharpest possible general "domination inequality" between the measures of dependence $\alpha_{1/p,1/p'}(\mathcal{F},\mathcal{G})$ and $R_{p,p'}(\mathcal{F},\mathcal{G})$ for any p, $1<p<\infty$, when one of the σ-fields is purely atomic with exactly two atoms, each having probability 1/2.

Remark 1.1. In Sections 2 and 4 we shall make use of the Lorentz spaces $\mathcal{L}_{p,q}=\mathcal{L}_{p,q}(\Omega,\mathcal{A},P)$, $1\leqslant p,q\leqslant\infty$. For details about Lorentz spaces and interpolation theory on them, the reader is referred to [1, 8, 19]. However, it will be worthwhile to briefly review a few of the most pertinent facts about them here. There is a very nice connection between the measures of dependence $\alpha_{r,s}$ and Lorentz spaces. For $1\leqslant p\leqslant\infty$ the Lorentz space $\mathcal{L}_{p,1}$ can be defined as the closure of the simple functions under the norm $\|f\|_{p,1}:=\inf\{\sum_i|a_i|\cdot P^{1/p}(A_i):\sum_i a_i I(A_i)=f\}$. (Thus $\mathcal{L}_{\infty,1}=\mathcal{L}_\infty$, and $\|f\|_{\infty,1}$ is within a fixed constant factor of $\|f\|_\infty$. Of course $I(\cdot)$ denotes the indicator function.) The following equations hold: For $1\leqslant p,q\leqslant\infty$.

$$\alpha_{1/p,1/q}(\mathcal{F},\mathcal{G})=\sup\frac{|Efg-EfEg|}{\|f\|_{p,1}\,\|g\|_{q,1}},\qquad f\in\mathcal{S}(\mathcal{F}),\ g\in\mathcal{S}(\mathcal{G}). \tag{1.3}$$

$$\forall f\in\mathcal{S}(\mathcal{A}),\ \|f\|_{1,1}=\|f\|_1. \tag{1.4}$$

$$\text{For } 1\leqslant p<q\leqslant\infty \text{ and } f\in\mathcal{S}(\mathcal{A}),\qquad \|f\|_{p,1}\leqslant C\,\|f\|_q, \tag{1.5}$$

where the constant C depends only on p and q. Equations (1.3) and (1.4) are easy to verify. To verify (1.5) it suffices to consider real nonnegative $f\in\mathcal{S}(\mathcal{A})$. The argument is well known: Construct a (nonnegative) r.v. f^* on the probability space [0, 1] (with Lebesgue measure) such that $f^*(t)$ is nonincreasing as t increases on $[0, 1]$ and f^* has the same distribution as f. Then

$$\|f\|_{p,1} \leqslant \int_0^1 [t^{-1/p'} f^*(t)]\, dt \leqslant \|t^{-1/p'}\|_{L_{q'}[0,1]} \|f^*\|_{\mathscr{L}_q[0,1]}. \tag{1.6}$$

To see the first inequality, it will be helpful to represent f by $\sum_{i=1}^{K} a_i I(A_i)$ with $A_1 \supset A_2 \supset \cdots \supset A_K$ and $a_i \geqslant 0$ and to represent f^* similarly. Of the two norms in the r.h.s. of the second inequality, the first is finite (assuming $p < q$) and depends only on p and q, and the second is $\|f\|_q$. Thus (1.5) holds.

II. The Pattern of Dominations

In the rest of this paper, a subscript or exponent of the form a_b will often be written $a(b)$ for typographical convenience.

Rosenblatt [13; 14, p. 211, Theorem 1] used the Riesz interpolation theorem to establish the equivalence of the measures of dependence $R_{p,p'}$, $1 < p < \infty$. From this result, the ones in [3], and the ones here in Section 2, it turns out that all of the nontrivial dominations between the measures $\alpha_{r,s}$ and $R_{p,q}$ in (1.1) and (1.2) are essentially consequences of interpolation theorems. In essence, interpolation theory determines precisely what dominations occur between these measures of dependence $\alpha_{r,s}$ and $R_{p,q}$.

For each (r, s) such that $0 < r, s \leqslant 1$ and $r + s > 1$, let $Q(r, s)$ denote the closed (convex) quadrilateral region with vertices $(0, 0)$, $(1, 0)$, (r, s), and $(0, 1)$. The main result of this section is as follows:

Proposition 2.1. *Statements* (a)–(h) *below give a complete list of all of the dominations between pairs of the measures of dependence* $\alpha_{r,s}$, $0 \leqslant r, s \leqslant 1$, *and* $R_{p,q}$, $1 \leqslant p, q \leqslant \infty$.

(a) *The measures of dependence* $\alpha_{r,s}$, $r + s < 1$, *and* $R_{p,q}$, $1/p + 1/q < 1$, *are equivalent. They do not dominate any of the other measures of dependence* $\alpha_{r,s}$, $R_{p,q}$.

(b) *The measures of dependence* $\alpha_{r,1-r}$, $0 < r < 1$, *and* $R_{p,p'}$, $1 < p < \infty$, *are equivalent. They dominate the measures* $\alpha_{r,s}$ *and* $R_{p,q}$ *specified in* (a) *but do not dominate any of the others.*

(c) *The measures of dependence* $\alpha_{1,0}$ *and* $R_{1,\infty}$ *are equivalent. They dominate the measures* $\alpha_{r,s}$ *and* $R_{p,q}$ *specified in* (a) *and* (b) *but do not dominate any of the others.*

(d) *The measures of dependence* $\alpha_{0,1}$ *and* $R_{\infty,1}$ *are equivalent. They dominate the measures* $\alpha_{r,s}$ *and* $R_{p,q}$ *specified in* (*a*) *and* (b) *but do not dominate any of the others.*

(e) *If* $0 < r_0, s_0 < 1$, *and* $r_0 + s_0 > 1$, *then the following three statements hold*:

(i) $\alpha_{r(0),s(0)}$ *and* $R_{1/r(0),1/s(0)}$ *each dominate* $\alpha_{r,s}$ *and* $R_{1/r,1/s}$ *for all* $(r,s) \in Q(r_0, s_0) - \{(1,0), (r_0, s_0), (0,1)\}$.

(ii) $R_{1/r(0),1/s(0)}$ *dominates* $\alpha_{r(0),s(0)}$, *but* $\alpha_{r(0),s(0)}$ *does not dominate* $R_{1/r(0),1/s(0)}$.

(iii) *Neither* $\alpha_{r(0),s(0)}$ *nor* $R_{1/r(0),1/s(0)}$ *dominates any of the measures of dependence* $\alpha_{1,0}$, $R_{1,\infty}$, $\alpha_{0,1}$, $R_{\infty,1}$ *or* $\alpha_{r,s}$ *or* $R_{1/r,1/s}$ *for* $(r,s) \notin Q(r_0, s_0)$.

(f) *If* $0 < r_0 < 1$ *and* $s_0 = 1$, *then statements* (i)–(iii) *in* (e) *all hold, except that* $\alpha_{r(0),1}$ *and* $R_{1/r(0),1}$ *each dominate* $\alpha_{0,1}$ *and* $R_{\infty,1}$.

(g) *If* $r_0 = 1$ *and* $0 < s_0 < 1$, *then statements* (i)–(iii) *in* (e) *all hold, except that* $\alpha_{1,s(0)}$ *and* $R_{1,1/s(0)}$ *each dominate* $\alpha_{1,0}$ *and* $R_{1,\infty}$.

(h) *The measures of dependence* $\alpha_{1,1}$ *and* $R_{1,1}$ *are equivalent* (*in fact, identical*), *and they dominate all of the other measures of dependence* $\alpha_{r,s}$ *and* $R_{1/r,1/s}$. (*End of Proposition* 2.1.)

The well-known measure of dependence associated with the "absolute regularity" [17] condition dominates only the measures $\alpha_{r,s}$ and $R_{p,q}$ in (a), and is dominated only by the ones $\alpha_{1,r}$, $\alpha_{r,1}$ $(0 \leqslant r \leqslant 1)$ and $R_{1,p}$, $R_{p,1}$ $(1 \leqslant p \leqslant \infty)$, see [5, Section 3].

The rest of this section is devoted to the proof of Proposition 2.1. Let us start by listing a few simple useful inequalities:

$$\alpha_{r,s}(\mathscr{F}, \mathscr{G}) \leqslant R_{1/r,1/s}(\mathscr{F}, \mathscr{G}), \tag{2.1}$$

$$\begin{gathered}\alpha_{r,s}(\mathscr{F}, \mathscr{G}) \text{ and } R_{1/r,1/s}(\mathscr{F}, \mathscr{G}) \text{ are each} \\ \text{nondecreasing as } r \text{ and/or } s \text{ increases,}\end{gathered} \tag{2.2}$$

$$\begin{aligned} &\alpha_{1,0}(\mathscr{F}, \mathscr{G}) \leqslant 1, \qquad R_{1,\infty}(\mathscr{F}, \mathscr{G}) \leqslant 2, \\ &\alpha_{0,1}(\mathscr{F}, \mathscr{G}) \leqslant 1. \qquad R_{\infty,1}(\mathscr{F}, \mathscr{G}) \leqslant 2. \end{aligned} \tag{2.3}$$

If $0 \leqslant r_0, r_1, s_0, s_1 \leqslant 1$, $0 < \theta < 1$, $r = (1-\theta)\, r_0 + \theta r_1$, and $s = (1-\theta)\, s_0 + \theta s_1$, then

$$\alpha_{r,s}(\mathscr{F}, \mathscr{G}) \leqslant [\alpha_{r(0),s(0)}(\mathscr{F}, \mathscr{G})]^{1-\theta}\, [\alpha_{r(1),s(1)}(\mathscr{F}, \mathscr{G})]^{\theta}, \tag{2.4}$$

$$R_{1/r,1/s}(\mathscr{F}, \mathscr{G}) \leqslant [R_{1/r(0),1/s(0)}(\mathscr{F}, \mathscr{G})]^{1-\theta}\, [R_{1/r(1),1/s(1)}(\mathscr{F}, \mathscr{G})]^{\theta}. \tag{2.5}$$

Equation (2.4) has a trivial one-line proof (used, e.g., in [3, Theorems 3.1 and 3.2]). Equation (2.5) is an application of Thorin's multilinear version of the Riesz–Thorin interpolation theorem (see, e.g., [1, p. 18, Exercise 13]). Note the analogous "structure" of (2.4) and (2.5).

The four equivalence classes of measures $\alpha_{r,s}$ and $R_{1/r,1/s}$, $r+s\leqslant 1$, specified in (a)–(d) of Proposition 2.1, are discussed in [3, Remark 4.1]. The pattern of dominations between them (as stated in (a)–(d)) is simply the well-known pattern of dominations between $\alpha_{0,0}$, $R_{2,2}$, $\alpha_{1,0}$, and $\alpha_{0,1}$. To complete the proof of (a)–(d) it just needs to be shown that none of those measures dominates any $\alpha_{r,s}$ or $R_{1/r,1/s}$ for $r+s>1$. If $r_0+s_0>1$ then for $\varepsilon>0$ sufficiently small, (r_0, s_0) would not be an element of either $Q(1,\varepsilon)$ or $Q(\varepsilon, 1)$. From (f) and (g), Eqs. (2.1) and (2.2), and the fact that "domination" is transitive, it would follow that none of the measures specified in (a)–(d) could dominate $\alpha_{r(0),s(0)}$ or $R_{1/r(0),1/s(0)}$. Thus, once (e)–(g) are proved, the proof of (a)–(d) would be complete. Also, (h) is well known and elementary. Thus, to complete the proof of Proposition 2.1, all that remains is to prove (e)–(g). Propositions 2.2–2.6 below are devoted to this purpose.

PROPOSITION 2.2. *Suppose that* $0\leqslant r_0, s_0, r_1, s_1\leqslant 1$, $r_0\neq r_1$, $s_0\neq s_1$, *and* $0<\theta<1$. *Define* r *and* s *by* $r:=(1-\theta)r_0+\theta r_1$ *and* $s:=(1-\theta)s_0+\theta s_1$. *Suppose that* $r+s>1$. *Then there exists a constant* $C=C(r_0, r_1, s_0, s_1, \theta)$ *such that for every pair of* σ*-fields* $\mathscr{F}$ *and* $\mathscr{G}$,

$$R_{1/r,1/s}(\mathscr{F},\mathscr{G})\leqslant C\cdot[\alpha_{r(0),s(0)}(\mathscr{F},\mathscr{G})]^{1-\theta}\cdot[\alpha_{r(1),s(1)}(\mathscr{F},\mathscr{G})]^{\theta}.$$

Proof. This proposition can be obtained by applying Zafran [19, p. 119, Theorem 2.9] to the bilinear form $B(f_1, f_2):=\operatorname{Cov}(f_1, f_2)$. To fit the hypothesis of that theorem, think of this bilinear form as a bilinear operator onto $\mathscr{L}_q(\Omega_0, \mathscr{H}_0, P_0)$, where q is arbitrary, Ω_0 is trivial, consisting of a single point, and $P_0(\Omega_0)=1$. First, a preliminary observation is needed. For $1\leqslant p<\infty$ and real nonnegative $f\in\mathscr{S}(\mathscr{H})$, by (1.6) one has $\|f\|_{p,1}\leqslant\|f\|^*_{p,1}$, where $\|f\|_{p,1}$ is as in Remark 1.1 and $\|f\|^*_{p,1}$ is as in [19, p. 108]. For $1\leqslant p\leqslant\infty$ and any $f\in\mathscr{S}(\mathscr{H})$, $\|f\|_{p,1}\leqslant 4\cdot\|f\|^*_{p,1}$ thus holds trivially. From this and (1.3) and an application of [19, Theorem 2.9], Proposition 2.2 follows. ∎

Remark. Proposition 2.2 also follows for $r+s=1$ as well as for $r+s>1$ by duality from the version of the Marcinkiewicz interpolation theorem given in Hunt [8, p. 264], applied to the linear operator $T(f):=E(f\mid\mathscr{G})-Ef$. But we shall use only the case $r+s>1$.

PROPOSITION 2.3. (i) *If* $0<r_0$, $s_0<1$ *and* $r_0+s_0>1$, *then* $\alpha_{r(0),s(0)}$ *dominates* $R_{1/r,1/s}$ *for every*

$$(r,s)\in Q(r_0,s_0)-\{(1,0),(r_0,s_0),(0,1)\}.$$

(ii) *If* $0 < r_0 < 1$ *then* $\alpha_{r(0),1}$ *dominates* $R_{1/r,1/s}$ *for every*

$$(r, s) \in Q(r_0, 1) - \{(1, 0), (r_0, 1)\}.$$

(iii) *If* $0 < s_0 < 1$ *then* $\alpha_{1,s(0)}$ *dominates* $R_{1/r,1/s}$ *for every*

$$(r, s) \in Q(1, s_0) - \{(1, s_0), (0, 1)\}.$$

Proof. We shall prove (i) first. Let S_1 denote the line segment with endpoints (r_0, s_0) and $(1, 0)$, and let S_2 denote the line segment with endpoints (r_0, s_0) and $(0, 1)$. By (2.3) and Proposition 2.2, $\alpha_{r(0),s(0)}$ dominates $R_{1/r,1/s}$ for every $(r, s) \in [S_1 \cup S_2] - \{(r_0, s_0), (1, 0), (0, 1)\}$. Also, by (2.2), $\alpha_{r(0),s(0)}$ dominates $\alpha_{0,0}$ and hence also $R_{1/r,1/s}$, $r+s<1$ (which are equivalent to $\alpha_{0,0}$ as noted in [3, Remark 4.1]). Finally, the remaining points (r, s) in the interior of $Q(r_0, s_0)$ each lie on some line segment with one endpoint on $[S_1 \cup S_2] - \{(r_0, s_0), (1, 0), (0, 1)\}$ and the other in $\{(r, s) : r+s<1\}$, and hence by (2.5), $\alpha_{r(0),s(0)}$ dominates $R_{1/r,1/s}$ for every such point (r, s). This completes the proof of (i).

To prove (ii), first note that $\alpha_{r(0),1}$ dominates $R_{1/r,1}$ for all r, $0 \leqslant r < r_0$, by (1.3), (1.4), and (1.5). The rest of the proof of (ii) is like that of (i). Finally, (iii) follows from (ii) by symmetry. ∎

Proposition 2.3 and Eq. (2.1) together give all of the dominations listed in parts (e)–(g) of Proposition 2.1. Now we are ready for the construction of counterexamples to prove the remaining, "negative" assertions in (e)–(g). A couple more trivial facts are worth keeping in mind. The quantity $|P(A \cap B) - P(A)\,P(B)|$ remains unchanged if A is replaced by A^c, or B by B^c; consequently, one always has

$$\alpha_{r,s}(\mathscr{F}, \mathscr{G}) = \sup \frac{|P(A \cap B) - P(A)\,P(B)|}{[P(A)]^r\,[P(B)]^s}, \qquad \begin{cases} A \in \mathscr{F},\ P(A) \leqslant \frac{1}{2}, \\ B \in \mathscr{G},\ P(B) \leqslant \frac{1}{2}. \end{cases}$$

Also, if $\mathscr{F}$ and $\mathscr{G}$ are finite σ-fields, each having exactly two atoms, then $R_{1/r,1/s}(\mathscr{F}, \mathscr{G}) \leqslant 4\alpha_{r,s}(\mathscr{F}, \mathscr{G})$ by a trivial argument.

PROPOSITION 2.4. (i) *If* $1 < p \leqslant \infty$ *then* $R_{p,1}$ *does not dominate* $\alpha_{1,0}$.

(ii) *If* $1 < q \leqslant \infty$ *then* $R_{1,q}$ *does not dominate* $\alpha_{0,1}$.

Proof. We shall first prove (i). For each ε, $0 < \varepsilon < \frac{1}{2}$, there exists a probability space and a pair of finite σ-fields $\mathscr{F} = \{\Omega, A, A^c, \phi\}$ and $\mathscr{G} = \{\Omega, B, B^c, \phi\}$ such that $P(A \cap B) = P(A) = \varepsilon$ and $P(B) = \frac{1}{2}$; and by a direct calculation, $R_{p,1}(\mathscr{F}, \mathscr{G}) \leqslant 4\alpha_{1/p,1}(\mathscr{F}, \mathscr{G}) = 4\varepsilon^{1/p'}$ and $\alpha_{1,0}(\mathscr{F}, \mathscr{G}) = \frac{1}{2}$. Statement (i) follows. Statement (ii) follows from (i) by symmetry. ∎

As a consequence of Proposition 2.4 (and Eq. (2.1)), if $r_0 < 1$, $s_0 < 1$, and $r_0 + s_0 > 1$, then $R_{1/r(0),\,1/s(0)}$ fails to dominate either $\alpha_{1,0}$, $R_{1,\infty}$, $\alpha_{0,1}$, or $R_{\infty,1}$. (If, for example, $R_{1/r(0),\,1/s(0)}$ were to dominate $\alpha_{1,0}$, then by (2.2) and "transitivity," $R_{1/r(0),\,1}$ would dominate $\alpha_{1,0}$, contradicting Proposition 2.4.) By (2.1), in order to complete the proof of (e)–(g) in Proposition 2.1 we only need to show that for (r_0, s_0) as in (e)–(g), $R_{1/r(0),\,1/s(0)}$ does not dominate $\alpha_{r,s}$ for any $(r, s) \notin Q(r_0, s_0)$ and $\alpha_{r(0),\,s(0)}$ does not dominate $R_{1/r(0),\,1/s(0)}$. These two facts will be shown respectively in Propositions 2.5 and 2.6 below.

PROPOSITION 2.5. *Suppose* $0 < r_0, s_0, r, s \leqslant 1$, $r_0 + s_0 > 1$, $r + s > 1$, *and* $(r, s) \notin Q(r_0, s_0)$. *Then* $R_{1/r(0),\,1/s(0)}$ *does not dominate* $\alpha_{r,s}$.

Proof. Let $ax + by = c$ be an equation of a line containing (r_0, s_0) and one of the points $(1, 0)$ or $(0, 1)$, such that the points $(0, 0)$ and (r, s) are in opposite half-planes determined by that line. By the assumptions in Proposition 2.5, we can (and do) take a, b, and c all positive. Thus $ar + bs > c$. Also (since $r_0 + s_0 > 1$) we have that $c = \max\{a, b\}$. Define $\varepsilon > 0$ by the equation $ar + bs = c + \varepsilon$.

For each n sufficiently large there exists a probability space and a pair of finite σ-fields $\mathscr{F} = \{\Omega, A, A^c, \phi\}$ and $\mathscr{G} = \{\Omega, B, B^c, \phi\}$ such that $P(A) = n^{-a} \leqslant \frac{1}{2}$, $P(B) = n^{-b} \leqslant \frac{1}{2}$, and $P(A \cap B) = n^{-a-b} + n^{-c-\varepsilon}$. For such an n it can easily be checked that $R_{1/r(0),\,1/s(0)}(\mathscr{F}, \mathscr{G}) \leqslant 4\alpha_{r(0),\,s(0)}(\mathscr{F}, \mathscr{G}) = 4n^{-\varepsilon}$ and that $\alpha_{r,s}(\mathscr{F}, \mathscr{G}) = 1$. Proposition 2.5 follows. ∎

PROPOSITION 2.6. *Suppose that* $0 < r, s \leqslant 1$, $r + s > 1$, *and either* $r < 1$ *or* $s < 1$. *Then* $\alpha_{r,s}$ *does not dominate* $R_{1/r,\,1/s}$.

Proof. By symmetry, without loss of generality we can (and do) assume that $r < 1$. (So we allow the possibility $s = 1$.)

Define the probability space $(\Omega, \mathscr{A}, P)$ as follows: $\Omega := [0, 1] \times \{0, 1\}$ (the union of two disjoint intervals); $\mathscr{A}$ is the σ-field of Borel subsets of Ω; and P is defined by

$$P(A \times \{0\}) := \int_A \Phi(x)\,dx$$

and (2.6)

$$P(A \times \{1\}) := \int_A (1 - \Phi(x))\,dx$$

for every Borel subset $A \subset [0, 1]$, where

$$\Phi(x) := \begin{cases} \kappa + \psi(x) & \text{if} \quad x \in [0, \frac{1}{2}] \\ \kappa - 2 \cdot \int_0^{1/2} \psi(u)\, du & \text{if} \quad x \in (\frac{1}{2}, 1] \end{cases} \tag{2.7}$$

where $0 < \kappa < \frac{1}{2}$ and

$$\psi(x) := \begin{cases} \varepsilon x^{r-1} & \text{if} \quad a < x < b \\ 0 & \text{otherwise}, \end{cases}$$

where

$$\varepsilon := \frac{\kappa^s}{(\log 1/\kappa)^{1-r}},$$

$$a := (\tfrac{1}{2})\, \kappa^{s/(1-r)},$$

$$b := (\tfrac{1}{2})\, \kappa^{(1-s)/r}.$$

Note that by our assumption $r + s > 1$ and simple arithmetic, we have that $0 < a < b \leqslant \frac{1}{2}$. Also, for κ sufficiently small, one has that $2 \cdot \int_0^{1/2} \psi(x)\, dx \leqslant \kappa$ and $\psi(x) \leqslant \frac{1}{2}$ for all $x \in [0, \frac{1}{2}]$, and hence $0 \leqslant \Phi(x) \leqslant 1$ for all $x \in [0, 1]$. Consequently, for κ sufficiently small, Eq. (2.6) does indeed define a probability measure. (We restrict κ to such small values.)

Define the "marginal" σ-fields $\mathscr{F}$ and $\mathscr{G}$ by

$$\mathscr{F} := \{A \times \{0, 1\} : A \subset [0, 1] \text{ Borel set}\},$$

$$\mathscr{G} := \{[0, 1] \times B : B = \{0, 1\}, \{0\}, \{1\}, \phi\}.$$

Define the event $B_0 := [0, 1] \times \{0\}$. Note that the marginal of P on $[0, 1]$ is Lebesgue measure and that $P(B_0) = \kappa < \frac{1}{2}$.

We shall first get an upper bound on $\alpha_{r,s}(\mathscr{F}, \mathscr{G})$. First a preliminary calculation will be handy. The function εx^{r-1} is non-increasing in $(0, \infty)$. Also, $\Phi(x) - \kappa$ is nonnegative for $0 \leqslant x \leqslant \frac{1}{2}$ and negative for $\frac{1}{2} < x \leqslant 1$. Hence, letting m denote Lebesgue measure, we have that for every Borel set $A \subset [0, 1]$,

$$\begin{aligned} \int_A [\Phi(x) - \kappa]\, dx &\leqslant \int_{A \cap [0, 1/2]} \psi(x)\, dx \leqslant \int_{A \cap [0, 1/2]} \varepsilon x^{r-1}\, dx \\ &\leqslant \int_0^{m(A \cap [0, 1/2])} \varepsilon x^{r-1}\, dx \\ &= (\varepsilon/r) \cdot [m(A \cap [0, 1/2])]^r \leqslant (\varepsilon/r) \cdot [m(A)]^r \end{aligned}$$

and

$$\begin{aligned}
\int_A [\Phi(x)-\kappa]\,dx &\geqslant -m(A\cap(\tfrac{1}{2},1])\cdot 2\cdot\int_0^{1/2}\psi(x)\,dx\\
&\geqslant -m(A\cap(\tfrac{1}{2},1])\cdot 2\cdot\int_0^{1/2}\varepsilon x^{r-1}\,dx\\
&= -(\varepsilon/r)\cdot m(A\cap(\tfrac{1}{2},1])\cdot(\tfrac{1}{2})^{r-1}\\
&\geqslant -(\varepsilon/r)\cdot m(A\cap(\tfrac{1}{2},1])\cdot[m(A\cap(\tfrac{1}{2},1])]^{r-1}\\
&\geqslant -(\varepsilon/r)\cdot[m(A)]^r
\end{aligned}$$

and hence $|\int_A [\Phi(x)-\kappa]\,dx| \leqslant (\varepsilon/r)\cdot[m(A)]^r$.

Consequently, it is easy to see that

$$\begin{aligned}
\alpha_{r,s}(\mathscr{F},\mathscr{G}) &= \sup\frac{|P(A_0\cap B_0)-P(A_0)\,P(B_0)|}{[P(A_0)]^r\cdot[P(B_0)]^s}, \qquad A_0\in\mathscr{F}\\
&= \sup\frac{|\int_A \Phi(x)\,dx-[m(A)]\cdot\kappa|}{[m(A)]^r\cdot\kappa^s}, \qquad A\subset[0,1]\text{ Borel set}\\
&\leqslant \sup\frac{(\varepsilon/r)\cdot[m(A)]^r}{[m(A)]^r\cdot\kappa^s}, \qquad A\subset[0,1]\text{ Borel set}\\
&= \frac{1}{r\cdot(\log 1/\kappa)^{1-r}}.
\end{aligned}$$

Note that $\alpha_{r,s}(\mathscr{F},\mathscr{G})$ becomes arbitrarily small as κ becomes sufficiently small. (We are using our assumption $r<1$ here.)

Now we only need to show that $R_{1/r,1/s}(\mathscr{F},\mathscr{G})$ fails to become small with κ. By a well-known, elementary result in functional analysis, we have for $p:=1/r$ and $q:=1/s$ (so that $1/p'=1-r$),

$$\begin{aligned}
R_{p,q}(\mathscr{F},\mathscr{G}) &= \sup\frac{\|E(g\,|\,\mathscr{F})-Eg\|_{p'}}{\|g\|_q}, \qquad g\in\mathscr{S}(\mathscr{G})\\
&\geqslant \frac{\|E(I(B_0)|\mathscr{F})-EI(B_0)\|_{p'}}{\|I(B_0)\|_q}\\
&= \frac{[\int_0^1|\Phi(x)-\kappa|^{p'}\,dx]^{1/p'}}{\kappa^s}\\
&\geqslant \frac{[\int_a^b(\psi(x))^{p'}\,dx]^{1/p'}}{\kappa^s}\\
&= \frac{[\int_a^b(\varepsilon\cdot x^{r-1})^{p'}\,dx]^{1-r}}{\kappa^s}
\end{aligned}$$

$$= \frac{\varepsilon \cdot [\int_a^b x^{-1}\, dx]^{1/r}}{\kappa^{-s}}$$

$$= \frac{1}{(\log 1/\kappa)^{1/r}} (\log b/a)^{1/r}$$

$$= \frac{1}{(\log 1/\kappa)^{1/r}} \cdot \left[\left(\frac{1-s}{r} - \frac{s}{1-r}\right)(\log \kappa)\right]^{1/r}$$

$$= \left(\frac{s}{1-r} - \frac{1-s}{r}\right)^{1/r} = \left(\frac{r+s-1}{r(1-r)}\right)^{1/r} > 0.$$

Hence $R_{p,q}(\mathscr{F}, \mathscr{G})$ fails to converge to 0 as $\kappa \to 0$. This completes the proof of Proposition 2.6 (and of Proposition 2.1). ▌

III. Sharpness of a Domination Result

This section deals with measures of dependence between an arbitrary finite number of σ-fields. Let us first recall some terminology from [3]. Suppose $(\Omega, \mathscr{M}, P)$ is a probability space, $n \geq 2$, and $\mathscr{F}_1, ..., \mathscr{F}_n$ are σ-fields $\subset \mathscr{M}$. Suppose $B: \mathscr{S}(\mathscr{F}_1) \times \cdots \times \mathscr{S}(\mathscr{F}_n) \to \mathbb{C}$ is an n-linear form, i.e., for each $i = 1, ..., n$, $B(f_1, ..., f_n)$ varies linearly with f_i (and $B(f_1, ..., f_n) = 0$ if $f_i = 0$ a.s.). Define the notation $[1, \infty]^n := \{(p_1, ..., p_n): 1 \leq p_k \leq \infty\ \forall k = 1, ..., n\}$. Suppose $\mathbf{p} := (p_1, ..., p_n) \in [1, \infty]^n$. As in [3] define

$$d_{\mathbf{p}}(B) := \sup \frac{|B(I(A_1), ..., I(A_n))|}{\|I(A_1)\|_{p(1)} \cdots \|I(A_n)\|_{p(n)}}, \qquad A_k \in \mathscr{F}_k \forall k = 1, ..., n;$$

$$\|B\|_{\mathbf{p}} := \sup \frac{|B(f_1, ..., f_n)|}{\|f_1\|_{p(1)} \cdots \|f_n\|_{p(n)}}, \qquad f_k \in \mathscr{S}(\mathscr{F}_k)\ \forall k = 1, ..., n.$$

The quantity $\|B\|_{\mathbf{p}}$ is the $\mathbf{p}$-norm of the n-linear form B, and $d_{\mathbf{p}}(B)$ is the corresponding "restricted" norm, the restriction being to indicator functions. In the special case where $n = 2$ and $B(f_1, f_2) := \operatorname{Cov}(f_1, f_2)$, we have $d_{\mathbf{p}}(B) = \alpha_{1/p(1), 1/p(2)}(\mathscr{F}_1, \mathscr{F}_2)$ and $\|B\|_{\mathbf{p}} = R_{p(1), p(2)}(\mathscr{F}_1, \mathscr{F}_2)$.

For each $k = 1, ..., n$ define the following vector $\in [1, \infty]^n$: $\mathbf{i}_k := (\infty, ..., \infty, 1, \infty, ..., \infty)$, where the 1 is the kth coordinate. The n-linear form B is said to be a "product form" if $\|B\|_{\mathbf{i}(k)} \leq 1\ \ \forall k = 1, ..., n$. By an application of Thorin's multilinear interpolation theorem (see, e.g., [3, p. 349, Theorem 3.3]), if B is a product form and $\mathbf{p} = (p_1, ..., p_n) \in [1, \infty]^n$ is such that $\sum_{k=1}^n 1/p_k \leq 1$, then $\|B\|_{\mathbf{p}} \leq 1$.

A key tool for the study of measures of dependence in [3] was the following theorem.

THEOREM A [3, Theorem 3.6]. *Suppose* $n \geqslant 2$. *Suppose* $\mathbf{p} := (p_1, ..., p_n) \in [1, \infty]^n$ *is such that* $\sum_{k=1}^{n} 1/p_k \leqslant 1$. *Define the number* $c = c(\mathbf{p}) := \sum_{\{k : p(k) < \infty\}} 1/p'_k$. *Then there exists a constant* $A = A(\mathbf{p})$ *which is a function only of* $\mathbf{p}$, *such that the following statement holds: If* $(\Omega, \mathscr{M}, P)$ *is a probability space,* $\mathscr{F}_1, ..., \mathscr{F}_n$ *are* σ*-fields* $\subset \mathscr{M}$, *and* $B: \mathscr{S}(\mathscr{F}_1) \times \cdots \times \mathscr{S}(\mathscr{F}_n) \to \mathbb{C}$ *is an* n*-linear product form, then* $\|B\|_{\mathbf{p}} \leqslant A \cdot d_{\mathbf{p}}(B) \cdot [1 - \log d_{\mathbf{p}}(B)]^c$.

Here $c := 0$ if $\mathbf{p} = (\infty, ..., \infty)$. The main result of this section is as follows:

THEOREM 3.1. *Suppose* $n \geqslant 2$, *and* $\mathbf{p} := (p_1, ..., p_n) \in [1, \infty]^n$. *Define the number* $c = c(\mathbf{p}) := \sum_{\{k : p(k) < \infty\}} 1/p'_k$. *Then there exists a positive constant* $a = a(\mathbf{p})$ *such that the following statement holds:*

For each t, $0 < t \leqslant 2^{-n}$, *there exists a probability space* $(\Omega, \mathscr{M}, P)$ *and* σ*-fields* $\mathscr{F}_1, ..., \mathscr{F}_n \subset \mathscr{M}$ *and an* n*-linear product form* $B: \mathscr{S}(\mathscr{F}_1) \times \cdots \times \mathscr{S}(\mathscr{F}_n) \to \mathbb{C}$ *(defined by* $B(f_1, ..., f_n) := E(f_1 \cdots f_n) - \prod_{k=1}^{n} Ef_k$*), such that* $d_{\mathbf{p}}(B) = t$ *and* $\|B\|_{\mathbf{p}} \geqslant a \cdot t(1 - \log t)^c$.

Remark 3.2. Several comments will be made:

(a) The assumption $\sum_{k=1}^{n} 1/p_k \leqslant 1$ in Theorem A is not required in Theorem 3.1.

(b) The constant $c = c(\mathbf{p})$ in Theorem 3.1 is exactly the same as in Theorem A. Consequently, Theorem 3.1 shows that Theorem A is within a constant factor of being sharp, for any choice of parameters meeting the specifications in Theorem A. (This "constant factor" may depend on the parameters.) Consequently [3, Theorem 4.1(vi)] is sharp in the same sense, by Theorem 3.1 for $n = 2$. Theorem 3.1 also shows indirectly that [3, Theorems 2.1 and 2.2] are sharp in the same sense; for if this were not so, then (see the proof of Theorem A) an improvement in [3, Theorems 2.1 and 2.2] (beyond just a better constant factor) would lead to a similar improvement in Theorem A, contradicting Theorem 3.1.

(c) The n-linear form B in Theorem 3.1 was chosen partly for its simplicity. Because of the extensive role played by cumulants in the study of dependence between more than two random variables, it is natural to consider measures of dependence based on norms of cumulants. For example, Mase [11] studied the measure of dependence $d_{(\infty, \infty, \infty, \infty)}(\mathrm{Cum})$ between four σ-fields, where Cum denotes the 4th-order cumulant. Theorem 3.1 holds with B defined by $B(f_1, ..., f_n) = \mathrm{Cum}(f_1, ..., f_n)$ (the nth-order cumulant). Because of our proof, this will be a trivial corollary of Theorem 3.1 itself; in our proof the construction will be such that any $n - 1$

of the σ-fields $\mathscr{F}_1, ..., \mathscr{F}_n$ are independent, and hence (after massive cancellation) one will have

$$\operatorname{Cum}(f_1, ..., f_n) = E(f_1 \cdot \cdots \cdot f_n) - \prod_{k=1}^{n} Ef_k \qquad \text{for} \quad f_k \in \mathscr{S}(\mathscr{F}_k),\ k = 1, ..., n.$$

For limit theory under mixing conditions involving other measures of dependence between three or more σ-fields, see e.g., Zurbenko [20], Statulevicius [15, 16], and Dmitrovskii *et al.* [7].

(d) One more elementary comment: Suppose that no restriction is imposed on the particular type of n-linear product form B. Then Theorem 3.1 can be extended trivially to the values t, $2^{-n} < t \leqslant 1$, by taking $\mathscr{F}_1 = \cdots = \mathscr{F}_n = \{\Omega, \phi\}$ and $B(f_1, ..., f_n) := tf_1 \cdot \cdots \cdot f_n$ (a constant) for $f_k \in \mathscr{S}(\mathscr{F}_k)$; and one can also extend Theorem 3.1 to the case $n = 1$ (for $0 < t \leqslant 1$). Theorem A also holds for $n = 1$. One can prove both theorems for $n = 1$ by a short direct argument or, alternatively, by converting the case $n = 1$, $1 \leqslant p_1 \leqslant \infty$, to the case $n = 2$, $\mathbf{p} = (p_1, \infty)$ in a trivial way. Of course the case $n = 1$ is not of much interest for the study of measures of dependence.

Proof of Theorem 3.1. The case where $p_k \in \{1, \infty\}$ $\forall k = 1, ..., n$ is very simple. To satisfy the requirements of both Theorem 3.1 and Remark 3.2(c), simply let $\Omega = \{-1, 1\}^n$, let $\mathscr{H}$ be the σ-field of all subsets of Ω, and define P by $P := (1 - \alpha) P_1 + \alpha P_2$ where $0 < \alpha \leqslant 1$ and P_1 and P_2 are the probability measures on $(\Omega, \mathscr{H})$ satisfying $P_1(\{z\}) = 2^{-n}$ $\forall z \in \Omega$ and $P_2(\{z\}) = 2^{-n+1}$ $\forall z := (z_1, ..., z_n) \in \Omega$ such that $z_1 \cdot \cdots \cdot z_n = 1$. Let $\mathscr{F}_1, ..., \mathscr{F}_n$ denote the coordinate σ-fields (each purely atomic with two atoms). Note that any $n - 1$ of them are independent. Consequently, in evaluating $d_{\mathbf{p}}(B)$, only *atoms* $A_k \in \mathscr{F}_k$ need to be considered. For the correct choice of α (depending on t), the n-linear form B defined in Theorem 3.1 will satisfy all requirements there (with $a = a(\mathbf{p}) := 1$). The details are elementary and are left to the reader.

To consider the remaining cases, henceforth we assume that $1 < p_k < \infty$ for at least one $k \in \{1, ..., n\}$.

First some preliminary calculations are needed. For each p, $1 < p < \infty$, and each v, $0 < v \leqslant \frac{1}{2}$, define the function $G_{v,p}$: $[0, 1] \to [0, 1]$ as follows:

$$G_{v,p}(x) := \min\{x, vx^{1/p}\} \qquad \text{for} \quad 0 \leqslant x \leqslant \tfrac{1}{2},$$

$$G_{v,p}(x) := G_{v,p}(1 - x) \qquad \text{for} \quad \tfrac{1}{2} \leqslant x \leqslant 1.$$

Note that G is concave and increasing on $[0, \frac{1}{2}]$ and hence G is concave on $[0, 1]$.

For $1<p<\infty$ and $0<v\leqslant\frac{1}{2}$ define the function $g_{v,p}\colon[0,1]\to[-1,1]$ by $g_{v,p}(x):=(d/dx)\,G_{v,p}(x)$. That is,

$$g_{v,p}(x):=\begin{cases}1 & \text{if } 0<x<v^{p'}\\ v\cdot(1/p)\cdot x^{-1/p'} & \text{if } v^{p'}<x<\frac{1}{2}\\ -v\cdot(1/p)\cdot(1-x)^{-1/p'} & \text{if } \frac{1}{2}<x<1-v^{p'}\\ -1 & \text{if } 1-v^{p'}<x<1\end{cases}\tag{3.1}$$

($g_{v,p}$ is not defined at $x=0$, $v^{p'}$, $\frac{1}{2}$, $1-v^{p'}$, 1.) Then $g_{v,p}$ is nonincreasing, $\int_0^1 g_{v,p}(x)\,dx=0$, and $|g_{v,p}(x)|\leqslant 1$ for all x at which $g_{v,p}(x)$ is defined.

If $0<v\leqslant\frac{1}{2}$ and $p=1$ or ∞, define the function $g_{v,p}\colon[0,1]\to[-1,1]$ by

$$g_{v,p}(x):=\begin{cases}1 & \text{if } 0<x<\frac{1}{2}\\ -1 & \text{if } \frac{1}{2}<x<1.\end{cases}\tag{3.2}$$

In this case too, $g_{v,p}$ is nonincreasing, $\int_0^1 g_{v,p}(x)\,dx=0$, and $|g_{v,p}(x)|\leqslant 1$.

The following integral will be used later on. If $1<p<\infty$ and $0<v\leqslant\frac{1}{2}$ then

$$\int_0^{1/2}[g_{v,p}(x)]^{p'}\,dx=v^{p'}[1+(1/p)^{p'}(\log\tfrac{1}{2}-p'\log v)].\tag{3.3}$$

Now let us get to the main part of the argument. Henceforth let m denote Lebesgue measure on $[0,1]$. Let t be arbitrary but fixed such that $0<t\leqslant 2^{-n}$ (as in Theorem 3.1). Define v, $0<v\leqslant\frac{1}{2}$, by

$$(\tfrac{1}{2})^{\operatorname{card}\{k:\,p(k)=\infty\}}\cdot v^{\operatorname{card}\{k:\,1<p(k)<\infty\}}=t.\tag{3.4}$$

Define the probability space $(\Omega,\mathscr{A},P)$ as follows: $\Omega:=[0,1]^n:=[0,1]\times[0,1]\times\cdots\times[0,1]$. $\mathscr{A}$ is the σ-field of Borel subsets of Ω. P is defined by

$$P(A_1\times\cdots\times A_n)=\prod_{k=1}^{n}m(A_k)+\prod_{k=1}^{n}\int_{A(k)}g_{v,p(k)}(x)\,dx\tag{3.5}$$

for all Borel subsets $A_1,\ldots,A_n\subset[0,1]$. (Recall the inequality $|g_{v,p}(x)|\leqslant 1$ mentioned above.)

Note that since $\int_0^1 g_{v,p}(x)=0$, each of the marginal distributions of P is uniform on $[0,1]$.

For each $k=1,\ldots,n$ let $\mathscr{F}_k$ be the σ-field generated by the kth coordinate in Ω. (Note that any $n-1$ of the σ-fields $\mathscr{F}_1,\ldots,\mathscr{F}_n$ are independent, as needed for Remark 3.2(c).) Define the n-linear form $B\colon\mathscr{S}(\mathscr{F}_1)\times\cdots\times$

$\mathscr{S}(\mathscr{F}_n) \to \mathbb{C}$ as in Theorem 3.1: $B(f_1, ..., f_n) := E(f_1 \cdot \cdots \cdot f_n) - \prod_{k=1}^{n} Ef_k$. As a consequence of (3.5) one has that for $f_k \in \mathscr{S}(\mathscr{F}_k)$, $k = 1, ..., n$,

$$B(f_1, ..., f_n) = \prod_{k=1}^{n} \int_0^1 f_k(x)\, g_{r,p(k)}(x)\, dx \tag{3.6}$$

(where $f_k(x_1, ..., x_{k-1}, x, x_{k+1}, ..., x_n)$ is written $f_k(x)$) and, since $|g_{r,p(k)}(x)| \leqslant 1$, it follows that B is a product form.

Proof that $d_{\mathbf{p}}(B) = t$. Suppose that $D_k \in \mathscr{F}_k$, $k = 1, ..., n$ with $P(D_k) \neq 0$. For each k, represent D_k by $D_k := [0, 1] \times \cdots \times [0, 1] \times B_k \times [0, 1] \times \cdots \times [0, 1]$ (where the kth coordinate set B_k is a Borel subset of $[0, 1]$). Then

$$\begin{aligned}
&|B(I(D_1), ..., I(D_n))| \\
&= \prod_{k=1}^{n} \left| \int_{B(k)} g_{r,p(k)}(x)\, dx \right| \\
&\leqslant \prod_{k=1}^{n} \left[\int_0^{m(B_k)} g_{r,p(k)}(x)\, dx \right] \\
&= \left[\prod_{\{k:\, 1 < p(k) < \infty\}} G_{r,p(k)}(m(B_k)) \right] \\
&\quad \times \left[\prod_{\{k:\, p(k) = 1 \text{ or } \infty\}} \left[\int_0^{m(B_k)} g_{r,p(k)}(x)\, dx \right] \right] \\
&\leqslant \left[\prod_{\{k:\, 1 < p(k) < \infty\}} r \cdot (m(B_k))^{1/p(k)} \right] \\
&\quad \times \left[\prod_{\{k:\, p(k) = 1\}} m(B_k) \right] \cdot \left[\prod_{\{k:\, p(k) = \infty\}} \tfrac{1}{2} \right] \\
&= (\tfrac{1}{2})^{\operatorname{card}\{k:\, p(k) = \infty\}} \cdot r^{\operatorname{card}\{k:\, 1 < p(k) < \infty\}} \cdot \prod_{k=1}^{n} \| I(D_k) \|_{p(k)} \\
&= t \cdot \prod_{k=1}^{n} \| I(D_k) \|_{p(k)}
\end{aligned} \tag{3.7}$$

by Eqs. (3.1), (3.2), and (3.4) and the fact that for each $0 < r \leqslant \frac{1}{2}$, $1 \leqslant p \leqslant \infty$, the function $g_{r,p}$ is nonincreasing (as noted earlier), odd-symmetric about $x = \frac{1}{2}$, and $\leqslant 1$. (In (3.7) of course $\prod_{\phi}$ (anything), the "empty product," is interpreted to be 1.) Thus $d_{\mathbf{p}}(B) \leqslant t$. To show that in fact $d_{\mathbf{p}}(B) = t$, note that in (3.7), equality is achieved in the case where $B_k = [0, \frac{1}{2}]$ $\forall k = 1, ..., n$.

Proof that $\|B\|_{\mathbf{p}} \geqslant a \cdot t(1 - \log t)^c$. Define the r.v.'s $f_1, ..., f_n$ as follows: $f_k(x_1, ..., x_n) := g_{r,p(k)}(x_k)$ if $p_k = 1$ or ∞ and $f_k(x_1, ..., x_n) := [\operatorname{sign} g_{r,p(k)}(x_k)] \cdot |g_{r,p(k)}(x_k)|^{p_k'/p_k}$ if $1 < p_k < \infty$. To shorten the notation below, we write $f_k(x)$ instead of $f_k(x_1, ..., x_{k-1}, x, x_{k+1}, ..., x_n)$. Note that for each k, $f_k \in \mathcal{L}_\infty(\mathcal{F}_k)$. (The fact that these r.v.'s are not all simple, is no problem. The n-linear form B extends uniquely to $\mathcal{L}_\infty(\mathcal{F}_1) \times \cdots \times \mathcal{L}_\infty(\mathcal{F}_n)$ without changing any norms.)

We shall use Eq. (3.6). For each k such that $p_k = 1$ or ∞,

$$\int_0^1 f_k(x)\, g_{r,p(k)}(x)\, dx = 1 \qquad \text{and} \qquad \| f_k \|_1 = \| f_k \|_\infty = 1.$$

For each k such that $1 < p_k < \infty$,

$$\int_0^1 f_k(x)\, g_{r,p(k)}(x)\, dx = 2 \cdot \int_0^{1/2} [g_{r,p(k)}(x)]^{p'_k}\, dx$$

and

$$\| f_k \|_{p(k)} = 2^{1/p(k)} \cdot \left[\int_0^{1/2} [g_{r,p(k)}(x)]^{p'_k}\, dx \right]^{1/p(k)}.$$

Hence

$$B(f_1, ..., f_n) = \left[\prod_{\{k:\, 1 < p(k) < \infty\}} 2^{1-1/p(k)} \times \left[\int_0^{1/2} [g_{r,p(k)}(x)]^{p'_k}\, dx \right]^{1-1/p(k)} \right] \prod_{k=1}^{n} \| f_k \|_{p(k)}.$$

By (3.3) and (3.4), for each k such that $1 < p_k < \infty$,

$$\int_0^{1/2} [g_{r,p(k)}(x)]^{p'_k}\, dx \geq a_k \cdot v^{p'_k}(1 - \log t),$$

where a_k is a positive constant that depends only on $\mathbf{p}$.

Hence by (3.4), $B(f_1, ..., f_n) \geq a \cdot t(1 - \log t)^c \cdot \prod_{k=1}^n \| f_k \|_{p(k)}$, where c is as in Theorem 3.1 and $a := \prod_{\{k:\, 1 < p(k) < \infty\}} a_k^{1/p'_k}$ (which is positive and depends only on $\mathbf{p}$). Thus $\| B \|_{\mathbf{p}} \geq a \cdot t(1 - \log t)^c$. This completes the proof of Theorem 3.1. ∎

IV. Application of the Reiteration Procedure

Except for a specific value for the constant factor, [3, Theorem 2.2] is as follows:

Theorem B. *Suppose $(\Omega, \mathcal{M}, P)$ is a probability space, $\mathcal{F}$ and $\mathcal{G}$ are σ-fields $\subset \mathcal{M}$, $1 < p < \infty$, and $0 < \varepsilon \leq 1$. Suppose $T: \mathcal{S}(\mathcal{F}) \to \mathcal{L}_\infty(\mathcal{G})$ is a linear operator such that*

$$\|T\|_{\mathscr{L}_1 \to \mathscr{L}_1} \leqslant 1, \tag{4.1}$$

$$\|T\|_{\mathscr{L}_\infty \to \mathscr{L}_\infty} \leqslant 1, \tag{4.2}$$

$$\|T\|_{\mathscr{L}_p \to \mathscr{L}_{p,\infty}} \leqslant \varepsilon. \tag{4.3}$$

Then

$$\|T\|_{\mathscr{L}_p \to \mathscr{L}_p} \leqslant C\varepsilon(1 - \log \varepsilon)^{1/p}, \tag{4.4}$$

where C is a constant that depends only on p.

In the case $T(f) := E(f \mid \mathscr{G}) - Ef$, one has the well-known connection $R_{p,q}(\mathscr{F}, \mathscr{G}) = \|T\|_{\mathscr{L}_p \to \mathscr{L}_q}$; and in [3, Theorem 4.1], Theorem B was applied to $\frac{1}{2}T$ for this T. The same type of connection in higher dimensions was a key tool in [3]. Of course, Theorem B can be generalized in several ways (for example, using infinite positive measures instead of probability measures).

The purpose of this section is to give a very short proof of Theorem B, using more interpolation theory than the proof in [3]. The proof here seems harder to generalize to the multidimensional case as in [3, Theorem 2.1]; to do this, one might use Zafran's [19, Theorem 2.9] multilinear Marcinkiewicz interpolation theorem (which we used in the proof of Proposition 2.2 above), but one first needs some bounds on the multiplicative constant in that theorem.

Proof of Theorem B. Throughout this proof, the constant C may vary from one appearance to the next, but it always depends only on p. Without loss of generality, we assume $0 < \varepsilon \leqslant e^{-2}$. The remaining cases either follow from this case or are trivial.

Let $\delta = -1/(\log \varepsilon)$. Then $0 < \delta \leqslant \frac{1}{2}$. Define p_0 and p_1 by $1/p_0 = (1-\delta)/p + \delta/1$ and $1/p_1 = (1-\delta)/p + \delta/\infty$. Then $1 < p_0 < p < p_1 < \infty$. Now we apply the Marcinkiewicz interpolation theorem twice, each time with an explicit upper bound on the constant in that theorem; see, e.g., [21, Chap. 12, Theorem (4.6) and Eq. (4.2.1)]. In that way, by (4.1) and (4.3) we obtain $\|T\|_{\mathscr{L}_{p(0)} \to \mathscr{L}_{p(0)}} \leqslant C \cdot (1/\delta)^{1/p(0)} \varepsilon^{1-\delta}$, and by (4.2) and (4.3) we obtain $\|T\|_{\mathscr{L}_{p(1)} \to \mathscr{L}_{p(1)}} \leqslant C \cdot (1/\delta)^{1/p(1)} \varepsilon^{1-\delta}$. Let $\theta = 1 - 1/p$. Then $0 < \theta < 1$ and $1/p = (1-\theta)/p_0 + \theta/p_1$. By applying the Riesz Thorin interpolation theorem (see [1, p. 9]) we have $\|T\|_{\mathscr{L}_p \to \mathscr{L}_p} \leqslant C \cdot (1/\delta)^{1/p} \varepsilon^{1-\delta} = C(-\log \varepsilon)^{1/p} \cdot \varepsilon \cdot \varepsilon^{-\delta} \leqslant C(-\log \varepsilon)^{1/p} \varepsilon$ (since $\varepsilon^{-\delta} = e$). This completes the proof. ∎

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