Slide 1

Quadratic harnesses, q-commutations, and orthogonal martingale polynomials

Włodek Bryc

University of Cincinnati

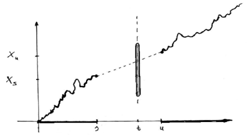
August 2005

Based on joint research with W. Matysiak and J. Wesolowski.

`math.uc.edu/~brycw/preprint/5-param/q-harnesses.pdf`

Slide 2

Notation and Motivation



$$\mathcal{F}_{s,u} = \sigma\{X_t : t \in (0,s] \cup [u,\infty)\}$$
$$\mathcal{F}_{\leq t} = \mathcal{F}_{t,\infty} = \sigma\{X_s : 0 < s \leq t\}$$

$0 < r < s < t < u < v$

Assumption: $E(X_t) = 0$ and $E(X_s X_t) = \min\{s,t\}$.

Slide 3

Quadratic harnesses

Covariance:

$$E(X_t) = 0,\ E(X_t X_s) = \min\{t,s\}. \tag{1}$$

Harness condition:

$$E(X_t | \mathcal{F}_{s,u}) = a_{t,s,u} X_s + b_{t,s,u} X_u, \tag{2}$$

Quadratic harness condition:

$$E\left(X_t^2 \middle| \mathcal{F}_{s,u}\right) = Q_{t,s,u}(X_s, X_u), \tag{3}$$

where

$$Q_{t,s,u}(x,y) = A_{t,s,u}x^2 + B_{t,s,u}xy + C_{t,s,u}y^2 + D_{t,s,u}x + E_{t,s,u}y + F_{t,s,u} \tag{4}$$

Note: Preserved by time-inversion $(X_t) \mapsto (tX_{1/t})$

Slide 4

Previous research

(i) Harnesses: [Hammersley, 1967] - linear regression based models of "long-range misorientation". [Williams, 1973] - discrete index, [1980] - continuous, [Mansuy and Yor, 2005] - integral repr.

(ii) [Plucińska, 1983]: linear regressions and constant conditional variances ⇒ Gaussian. Discrete indexes [Bryc and Plucińska, 1985], L_2-smooth processes [Szabłowski, 1989], Poisson [Bryc, 1987], Gamma [Wesołowski, 1989]

(iii) [Wesołowski, 1993]: conditional variance as a quadratic function of increments characterizes five Lévy processes: Wiener, Poisson, Pascal (neg. bin.), Gamma, and Meixner (hyp. secant).

(iv) [Bryc, 2001b]: stationary "quadratic harnesses" are classical versions of non-commutative q-Gaussian processes of [Frisch and Bourret, 1970] and [Bożejko et al., 1997].

Slide 5

Technical Assumptions:

(i) $Q_{t,s,u}(0,0) = F_{t,s,u} \neq 0$

(ii) $1, X_s, X_t, X_sX_t, X_s^2, X_t^2$ are linearly independent for all $0 < s < t$.

Theorem 1 ([Bryc et al., 2005]) *There exist $\eta, \theta \in \mathbb{R}$, $\sigma, \tau \geq 0$, and $q \leq 1 + 2\sqrt{\sigma\tau}$ such that*

$$\mathrm{Var}\left(\mathrm{X_t} \middle| \mathcal{F}_{\mathrm{s,u}}\right) = F_{t,s,u} K\left(\frac{X_u - X_s}{u-s}, \frac{uX_s - sX_u}{u-s}\right) \qquad (5)$$

for all $0 < s < t < u$, where $F_{t,s,u} = \frac{(u-t)(t-s)}{u(1+\sigma s)+\tau - qs}$ is the normalizing constant, and

$$K(\mathsf{x}, \mathsf{y}) = 1 + \theta \mathsf{x} + \tau \mathsf{x}^2 + \eta \mathsf{y} + \sigma \mathsf{y}^2 - (\mathsf{xy} - q\mathsf{yx})$$

Notation: (X_t) is a quadratic harness with parameters $q, \eta, \theta, \sigma, \tau$

Slide 6

Martingale polynomials

$$E\left(p_n\left(X_t; t\right) \middle| \mathcal{F}_{\leq s}\right) = p_n\left(X_s; s\right),\ 0 < s < t,\ n = 0, 1, \ldots.$$

$$x p_n(x;t) = \sum_{k=0}^{n+1} C_{k,n}(t) p_k(x;t).$$

Take $p_0 = 1$, $p_1 = x$. Then

$$\mathbf{C}(t) := \begin{bmatrix} 0 & t & C_{02}(t) & C_{0,3}(t) & \ldots & \\ 1 & C_{11}(t) & C_{12}(t) & C_{13}(t) & \ldots & \\ 0 & C_{21}(t) & C_{22}(t) & C_{23}(t) & \ldots & \\ 0 & 0 & C_{32}(t) & C_{33}(t) & \ldots & \\ 0 & 0 & 0 & C_{43}(t) & \ddots & \\ \vdots & \vdots & \vdots & & & \ddots \end{bmatrix}$$

Slide 7

Suppose for each $t > 0$ the random variable X_t has infinite support.

Theorem 2 *If a quadratic harness (X_t) with parameters $q, \eta, \theta, \sigma, \tau$ has finite moments of all orders and martingale polynomials, then*

$$\mathbf{C}(t) = t\mathsf{x} + \mathsf{y},\ t > 0, \qquad (6)$$

and the infinite matrices $\mathsf{x} = \mathbf{C}_1 - \mathbf{C}_0$, $\mathsf{y} = \mathbf{C}_0$ satisfy the q-commuation equation $K(\mathsf{x}, \mathsf{y}) = 0$, i.e.

$$[\mathsf{x}, \mathsf{y}]_q = I + \theta \mathsf{x} + \eta \mathsf{y} + \tau \mathsf{x}^2 + \sigma \mathsf{y}^2, \qquad (7)$$

where

$$[\mathsf{x}, \mathsf{y}]_q = \mathsf{xy} - q\mathsf{yx},$$

In [Frisch and Bourret, 1970], the equation $[\mathsf{x}, \mathsf{x}^*]_q = I$ is the basis of a "parastochastic" model of Kraichnan's equation (Guionnett-Mazza 2004). [Bryc and Wesołowski, 2005]: quadratic harnesses with $\eta = \theta = \tau = \sigma = 0$ are a classical version of the q-Brownian motion.

Slide 8

Orthogonal martingale polynomials

$$\mathbf{C}(t) := \begin{bmatrix} 0 & t & 0 & 0 & \ldots & \\ 1 & \gamma_1 t + \delta_1 & \varepsilon_2 t + \varphi_2 & 0 & \ldots & \\ 0 & \alpha_2 t + \beta_2 & \gamma_2 t + \delta_2 & \varepsilon_3 t + \varphi_3 & \ldots & \\ 0 & 0 & \alpha_3 t + \beta_3 & \gamma_3 t + \delta_3 & \ldots & \\ 0 & 0 & 0 & \alpha_4 t + \beta_4 & \ddots & \\ \vdots & \vdots & \vdots & & & \ddots \end{bmatrix}$$

Slide 9

Orthogonal martingale polynomials

$$xp_n(x;t) = a_n(t)p_{n+1}(x;t) + b_n(t)p_n(x;t) + c_n(t)p_{n-1}(x;t), \quad (8)$$

$$a_n(t) = \alpha_{n+1}t + \beta_{n+1}\ , \quad b_n(t) = \gamma_n t + \delta_n\ , \quad c_n(t) = \varepsilon_n t + \varphi_n, \quad (9)$$

and the coefficients in (9) satisfy a system of 5 equations that result from the q-commutation equation (7).

Theorem 3 ([Bryc et al., 2005]) *Suppose (X_t) is a quadratic harness with parameters such that $0 \le \sigma\tau < 1$, $-1 < q \le 1 - 2\sqrt{\sigma\tau}$. Moreover, assume that for each $t > 0$ random variable X_t has moments of all orders and infinite support. Then (8) determines $\{p_n(x;t)\}$ which are orthogonal martingale polynomials for (X_t).*

Slide 10

$$a_n(t) = \sigma\alpha_{n+1}t + \beta_{n+1}\ , \quad b_n(t) = \gamma_n t + \delta_n\ , \quad c_n(t) = (\beta_n t + \tau\alpha_n)\,\omega_n\ , \quad (10)$$

(i) $\alpha_1 = 0,\ \beta_1 = 1,\ \gamma_0 = \delta_0 = 0,\ \omega_1 = 1.$

(ii) $\begin{bmatrix} \alpha_{n+1} \\ \beta_{n+1} \end{bmatrix} = \begin{bmatrix} q & 1 \\ -\sigma\tau & 1 \end{bmatrix} \times \begin{bmatrix} \alpha_n \\ \beta_n \end{bmatrix}, \quad n \ge 1.$

Moreover, $\lambda_{n,k} := \beta_n\beta_{n+k} - \sigma\tau\alpha_n\alpha_{n+k} > 0$ for all $n \ge 1$, $k \ge 0$.

(iii)

$$\gamma_{n+1} = \frac{q+\sigma\tau}{\lambda_{n+2,0}}\left(\lambda_{n,2}\gamma_n + (\alpha_{n+2}\beta_n - \beta_{n+2}\alpha_n)\sigma\delta_n\right) + \frac{\sigma\alpha_{n+2}}{\lambda_{n+2,0}}(\eta\tau\alpha_{n+1} + \theta\beta_{n+1}) + \frac{\beta_{n+2}}{\lambda_{n+2,0}}(\theta\sigma\alpha_{n+1} + \eta\beta_{n+1})$$

Similar equation for δ_{n+1} and ω_n.

Slide 11

Worked out Example I: $[\mathbf{x},\mathbf{y}]_q = \mathbf{I} + \theta\mathbf{x} + \tau\mathbf{x}^2$

$$\mathsf{x} = \mathsf{D}_q$$
$$\mathsf{y} = \mathsf{Z}(1 + \theta\mathsf{D}_q + \tau\mathsf{D}_q^2)$$

where

$$\mathsf{D}_q(f)(z) = \frac{f(z) - f(qz)}{(1-q)z}, \ \mathsf{Z}(f)(z) = zf(z),$$

Proof:

$$[\mathsf{x},\mathsf{y}]_q = [\mathsf{D}_q,\mathsf{Z}]_q + \theta[\mathsf{D}_q,\mathsf{Z}\mathsf{D}_q]_q + \tau[\mathsf{D}_q,\mathsf{Z}\mathsf{D}_q^2]_q = \mathsf{I} + \theta\mathsf{D}_q + \tau\mathsf{D}_q^2.$$

Note: $\mathsf{D}_q(z^n) = [n]_q z^{n-1}$, where $[n]_q = 1 + q + \ldots + q^{n-1}$.
Calculation:

$$(t\mathsf{x} + \mathsf{y})z^n = z^{n+1} + \theta[n]_q z^n + (t + \tau[n-1]_q)[n]_q z^{n-1}.$$

Slide 12

Identification of z^n with $p_n(x;t)$ gives Al-Salam-Chihara polynomials

$$xp_n(x;t) = p_{n+1}(x;t) + \theta[n]_q p_n(x;t) + (t + \tau[n-1]_q)[n]_q p_{n-1}(x;t).$$

[Feinsilver, 1990, Section 3.4], [Anshelevich, 2003, Remark 6].

Theorem 4 ([Bryc and Wesołowski, 2005]) *If (X_t) is a quadratic harness with parameters $-1 < q \le 1$, $\sigma = \eta = 0$ then (X_t) is Markov with the transition probabilities $P_{s,t}(x,dy)$ determined as the unique probability measure orthogonalizing polynomials $\{Q_n(y)\}$ given by the recurrence*

$$yQ_n(y)$$
$$= Q_{n+1}(y) + (\theta[n]_q + xq^n)Q_n(y) + (t - sq^{n-1} + \tau[n-1]_q)[n]_q Q_{n-1}(y).$$

Conversely, each such Markov process is a quadratic harness.

Slide 13

Classical versions $[\mathsf{x},\mathsf{y}]_q = \mathsf{I} + \theta\mathsf{x} + \tau\mathsf{x}^2$

$$E(X_{t_1}X_{t_2}\dots X_{t_k}) = \tau(\mathbf{X}_{t_1}\mathbf{X}_{t_2}\dots\mathbf{X}_{t_k}),\ \forall_{t_1\le t_2\le\dots\le t_k}$$

$$\mathrm{Var}(X_t|\mathcal{F}_{s,u}) \asymp 1+\theta\frac{X_u - X_s}{u-s}+\tau\frac{(X_u-X_s)^2}{(u-s)^2}-(1-q)\frac{(X_u-X_s)(uX_s-sX_u)}{(u-s)^2}$$

(i) Case $\tau=\theta=0$ and $-1<q\le 1$ corresponds to the q-Brownian motion of [Bożejko et al., 1997].

(ii) Case $\theta\ne 0$, $\tau=0$ and $-1<q\le 1$ correspond to the q-Poisson process of [Anshelevich, 2001]. It is also a classical version of the time-inverse of the q-Poisson process of [Saitoh and Yoshida, 2000].

(iii) If $\tau\ne 0, \theta\ne 0$, $\sigma=\eta=0$ and $-1<q<1$, $q\ne 0$ then the the laws of q-Meixner process (X_t) are not closed under q-convolution of [Nica, 1995], and are not the laws of the q-Levy processes of [Anshelevich, 2004].

Slide 14

Worked Out Example II: $[\mathsf{x},\mathsf{y}]_q = \mathsf{I} + \theta\mathsf{x} + \eta\mathsf{y}$

Operator solution:

$$\mathsf{x} = \mathsf{D}_q + \eta\mathsf{Z}(\mathsf{D}_q + \theta\mathsf{D}_q^2),\ \mathsf{y} = \mathsf{Z}(1+\theta\mathsf{D}_q)$$

Calculation:

$$(t\mathsf{x}+\mathsf{y})z^n = z^{n+1} + (\theta+t\eta)[n]_q z^n + t(1+\eta\theta[n-1]_q)[n]_q z^{n-1},$$

These are **again** Al-Salam-Chihara polynomials

$$xp_n(x;t) = p_{n+1}(x;t)+(\theta+t\eta)[n]_q p_n(x;t)+t(1+\eta\theta[n-1]_q)[n]_q p_{n-1}(x;t), \quad (11)$$

Note: We must have $1+\eta\theta\ge q^+$.

Slide 15

Theorem 5 (work in progress) *If (X_t) is a quadratic harness with parameters $-1<q\le 1$, $\sigma=\tau=0$, and $1+\eta\theta>\max\{q,0\}$ then (X_t) is Markov with the transition probabilities $P_{s,t}(x,dy)$ determined as the unique probability measure orthogonalizing polynomials $\{Q_n(y)\}$ given by the recurrence*

$$yQ_n(y) = Q_{n+1}(y) + \mathcal{A}_n(x,t,s)Q_n(y) + \mathcal{B}_n(x,t,s)Q_{n-1}(y),$$

where

$$\mathcal{A}_n(x,t,s) = q^n x + [n]_q(t\eta+\theta-[2]_q q^{n-1}s\eta),$$

$$\mathcal{B}_n(x,t,s) = [n]_q(t-sq^{n-1})\left(1+\eta x q^{n-1} + [n-1]_q\eta(\theta - s\eta q^{n-1})\right).$$

Conversely, each such Markov process is a quadratic harness.

Slide 16

Free case: $q=0$

$$\mathrm{Var}(X_t|\mathcal{F}_{s,u}) \asymp 1+\theta\frac{X_u-X_s}{u-s}+\eta\frac{uX_s-sX_u}{u-s}-\frac{(X_u-X_s)(uX_s-sX_u)}{(u-s)^2}$$

Denote by π_t the law of X_t.

Proposition 6 ([Bryc and Wesołowski, 2004]) *For every $t\ge 0$, there exist a probability measure ν_t such that the pairs (π_t,ν_t) form a semigroup with respect to the c-convolution,*

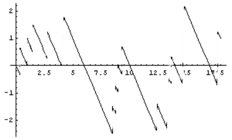
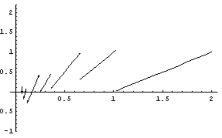
$$(\pi_{t+s},\nu_{t+s}) = (\pi_t,\nu_t)\bigstar_c(\pi_s,\nu_s).$$

[Bożejko et al., 1996], [Bożejko and Wysoczański, 2001], [Krystek and Wojakowski, 2004].

Slide 17

Classical case: $q = 1$

$$\mathrm{Var}(\mathrm{X_t}|\mathcal{F}_{\mathrm{s,u}}) \asymp 1 + \theta \frac{\mathrm{X_u} - \mathrm{X_s}}{\mathrm{u} - \mathrm{s}} + \eta \frac{\mathrm{uX_s} - \mathrm{sX_u}}{\mathrm{u} - \mathrm{s}}$$

$\mathrm{Var}(\mathrm{X_t}\|\mathcal{F}_{\mathrm{s,u}}) \asymp 1 + \theta \frac{\mathrm{X_u}-\mathrm{X_s}}{\mathrm{u}-\mathrm{s}}$	$\mathrm{Var}(\mathrm{X_t}\|\mathcal{F}_{\mathrm{s,u}}) \asymp 1 + \eta \frac{\mathrm{uX_s}-\mathrm{sX_u}}{\mathrm{u}-\mathrm{s}}$
Poisson type	time-inverse of Poisson
$X_t = N_t - t$	$X_t = tN_{1/t} - 1$
	

Slide 18

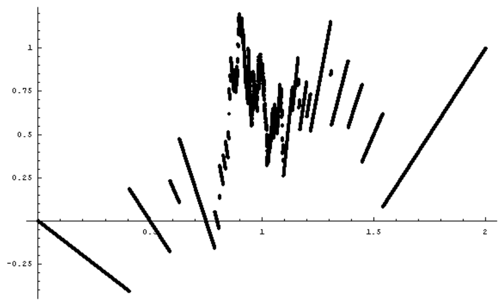
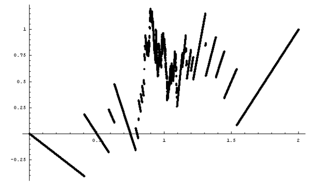


Figure 1: Simulated sample trajectory when $\eta = \theta$.

Slide 19

Birth-then-death process



$$X_t = \begin{cases} (\theta - \eta t) Z_t - \frac{t}{\theta} & 0 < t < T, \\ Z_T - \frac{1}{\eta} & t = T \\ (\eta t - \theta) Z_t - \frac{1}{\eta} & t > T \end{cases}$$

Here $T = \eta/\theta$

Slide 20

Fix $\eta, \theta > 0$. Let $T = \theta/\eta$. Let $(Z_t)_{t>0}$ be a non-homogeneous Markov process with rates

$k \mapsto k + 1$ at rate $(k + \frac{1}{\eta\theta})/(T - s)$ on $0 < s < t < T$

$k \mapsto k - 1$ at rate $k/(s - T)$ on $T < s < t$

$$Z_t - Z_s | Z_s \stackrel{d}{=} nb\left(\frac{1}{\theta\eta} + Z_s, \frac{\theta - \eta t}{\theta - \eta s}\right), \quad 0 \le s < t < \theta/\eta,$$

$$Z_T | Z_s \stackrel{d}{=} \mathrm{Gamma}\left(\frac{1}{\theta\eta} + Z_s, \frac{1}{\theta - \eta s}\right), \quad 0 \le s < T,$$

$$Z_t | Z_T \stackrel{d}{=} \mathcal{P}\left(\frac{Z_{\theta/\eta}}{\eta t - \theta}\right), \quad t > T,$$

$$Z_t | Z_s \stackrel{d}{=} b\left(Z_s, \frac{\eta s - \theta}{\eta t - \theta}\right), T < s < t.$$

Slide 21

Concluding Remarks

(i) Little is known about the non-commutative processes corresponding to $[\mathsf{x}, \mathsf{y}]_q = \mathsf{I} + \theta\mathsf{x} + \eta\mathsf{y}$.

(ii) Little is known about which q-Guassian processes have classical versions. Markov, yes [Bożejko et al., 1997]. There are q-Gaussian processes with no classical version [Bryc, 2001a].

(iii) No general constructions of quadratic harnesses are known.

Slide 22

References

[Anshelevich, 2001] Anshelevich, M. (2001). Partition-dependent stochastic measures and q-deformed cumulants. *Doc. Math.*, 6:343–384 (electronic).

[Anshelevich, 2003] Anshelevich, M. (2003). Free martingale polynomials. *Journal of Functional Analysis*, 201:228–261. ArXiv:math.CO/0112194.

[Anshelevich, 2004] Anshelevich, M. (2004). q-Lévy processes. *J. Reine Angew. Math.*, 576:181–207. ArXiv:math.OA/03094147.

[Biane, 1998] Biane, P. (1998). Processes with free increments. *Math. Z.*, 227(1):143–174.

[Bożejko et al., 1997] Bożejko, M., Kümmerer, B., and Speicher, R. (1997). q-Gaussian processes: non-commutative and classical aspects. *Comm. Math. Phys.*, 185(1):129–154.

[Bożejko et al., 1996] Bożejko, M., Leinert, M., and Speicher, R. (1996). Convolution and limit theorems for conditionally free random variables. *Pacific J. Math.*, 175(2):357–388.

[Bożejko and Wysoczański, 2001] Bożejko, M. and Wysoczański, J. (2001). Remarks on t-transformations of measures and convolutions. *Ann. Inst. H. Poincaré Probab. Statist.*, 37(6):737–761.

[Bryc, 1987] Bryc, W. (1987). A characterization of the Poisson process by conditional moments. *Stochastics*, 20:17–26.

[Bryc, 2001a] Bryc, W. (2001a). Classical versions of q-Gaussian processes: conditional moments and Bell's inequality. *Comm. Math. Physics*, 219:259–270.

[Bryc, 2001b] Bryc, W. (2001b). Stationary random fields with linear regressions. *Annals of Probability*, 29:504–519.

[Bryc et al., 2005] Bryc, W., Matysiak, W., and Wesołowski, J. (2005). Quadratic harnesses, q-commuttions, and orthogonal martingale polynomials. arxiv.org/abs/math.PR/0504194.

[Bryc and Plucińska, 1985] Bryc, W. and Plucińska, A. (1985). A characterization of infinite gaussian sequences by conditional moments. *Sankhya A*, 47:166–173.

Slide 23

[Bryc and Wesołowski, 2004] Bryc, W. and Wesołowski, J. (2004). Bi-Poisson process. Submitted. arxiv.org/abs/math.PR/0404241.

[Bryc and Wesołowski, 2005] Bryc, W. and Wesołowski, J. (2005). Conditional moments of q-Meixner processes. *Probability Theory Related Fields*, 131:415–441. arxiv.org/abs/math.PR/0403016.

[Feinsilver, 1990] Feinsilver, P. (1990). Lie algebras and recurrence relations. III. q-analogs and quantized algebras. *Acta Appl. Math.*, 19(3):207–251.

[Frisch and Bourret, 1970] Frisch, U. and Bourret, R. (1970). Parastochastics. *J. Math. Phys.*, 11(2):364–390.

[Hammersley, 1967] Hammersley, J. M. (1967). Harnesses. In *Proc. Fifth Berkeley Sympos. Mathematical Statistics and Probability (Berkeley, Calif., 1965/66), Vol. III: Physical Sciences*, pages 89–117. Univ. California Press, Berkeley, Calif.

[Krystek and Wojakowski, 2004] Krystek, A. and Wojakowski, Ł. (2004). Associative convolutions arising from conditionally free convolution. Preprint.

[Mansuy and Yor, 2005] Mansuy, R. and Yor, M. (2005). Harnesses, Lévy bridges and Monsieur Jourdain. *Stochastic Processes and Their Applications*, 115:329–338.

[Meixner, 1934] Meixner, J. (1934). Orthogonale polynomsysteme mit einer besonderen gestalt der erzeugenden funktion. *Journal of the London Mathematical Society*, 9:6–13.

[Nica, 1995] Nica, A. (1995). A one-parameter family of transforms, linearizing convolution laws for probability distributions. *Comm. Math. Phys.*, 168(1):187–207.

[Plucińska, 1983] Plucińska, A. (1983). On a stochastic process determined by the conditional expectation and the conditional variance. *Stochastics*, 10:115–129.

[Saitoh and Yoshida, 2000] Saitoh, N. and Yoshida, H. (2000). q-deformed Poisson random variables on q-Fock space. *J. Math. Phys.*, 41(8):5767–5772.

[Szabłowski, 1989] Szabłowski, P. J. (1989). Can the first two conditional moments identify a mean square differentiable process? *Comput. Math. Appl.*, 18(4):329–348.

[Voiculescu, 2000] Voiculescu, D. (2000). Lectures on free probability theory. In *Lectures on probability

Slide 24

theory and statistics (Saint-Flour, 1998), volume 1738 of *Lecture Notes in Math.*, pages 279–349. Springer, Berlin.

[Wesołowski, 1989] Wesołowski, J. (1989). A characterization of the gamma process by conditional moments. *Metrika*, 36(5):299–309.

[Wesołowski, 1993] Wesołowski, J. (1993). Stochastic processes with linear conditional expectation and quadratic conditional variance. *Probab. Math. Statist.*, 14:33–44.

[Williams, 1973] Williams, D. (1973). Some basic theorems on harnesses. In *Stochastic analysis (a tribute to the memory of Rollo Davidson)*, pages 349–363. Wiley, London.