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# $q$-Gaussian Processes: Non-commutative and Classical Aspects

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**Abstract:** We examine, for $-1 < q < 1$, $q$-Gaussian processes, i.e. families of operators (non-commutative random variables) $X_t = a_t + a_t^*$ – where the $a_t$ fulfill the $q$-commutation relations $a_s a_t^* - q a_t^* a_s = c(s,t) \cdot \mathbf{1}$ for some covariance function $c(\cdot,\cdot)$ – equipped with the vacuum expectation state. We show that there is a $q$-analogue of the Gaussian functor of second quantization behind these processes and that this structure can be used to translate questions on $q$-Gaussian processes into corresponding (and much simpler) questions in the underlying Hilbert space. In particular, we use this idea to show that a large class of $q$-Gaussian processes possesses a non-commutative kind of Markov property, which ensures that there exist classical versions of these non-commutative processes. This answers an old question of Frisch and Bourret [FB].

## Introduction

What we are going to call $q$-Gaussian processes was essentially introduced in a remarkable paper by Frisch and Bourret [FB]. Namely, they considered generalized commutation relations given by operators $A(t)$ and a vacuum vector $\Psi_0$ with

$$A(t)A^*(t') - qA^*(t')A(t) = \Gamma(t,t')\mathbf{1}$$

and

$$A(t)\,\Psi_0 = 0$$

for some real covariance function $\Gamma$ (i.e. positive definite function). The aim of the authors was to study the probabilistic properties of the "parastochastic" process $M(t) = A(t) + A^*(t)$.

The basic problems arising in this context were the following two types of questions:

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(I) (realization problem)
Do there exist operators on some Hilbert space and a corresponding vacuum vector in this Hilbert space which fulfill the above relations, i.e. are there non-commutative realizations of the $q$-Gaussian processes?

(II) (random representation problem)
Are these non-commutative processes of a classical relevance, i.e. do there exist classical versions of the $q$-Gaussian processes (in the sense of coinciding time-ordered correlations, see our Definition 4.1)?

Frisch and Bourret could give the following partial answers to these questions.

(I) For $q = \pm 1$ the realization is of course given by the Fock space realization of the bosonic/fermionic relations. The case $q = 0$ was realized by creation and annihilation operators on the full Fock space (note that this was before the introduction of the Cuntz algebras and their extensions [Cun, Eva]). For other values of $q$ the realization problem remained open.

(II) The $q = 1$ processes are nothing but the Fock space representations of the classical Gaussian processes. For $q = -1$ a classical realization by a dichotomic Markov process could be given for the special case of exponential covariance $\Gamma(t, t') = \exp(-|t - t'|)$. A classical realization for $q = 0$ could not be found, but the authors were able to show that there is an interesting representation in terms of Gaussian random matrices.

The authors started also the investigation of parastochastic equations (i.e. the coupling of parastochastic processes to other systems), but – probably because of the open problem on the mere existence and classical relevance of these $q$-processes – there was apparently no further work in this direction and the paper of Frisch and Bourret fell into oblivion.

Starting with [AFL] there has been another and independent approach to non-commutative probability theory. This wide and quite inhomogenous field – let us just mention as two highlights the quantum stochastic calculus of Hudson-Parthasarathy [HP] and the free probability theory of Voiculescu [VDN] – is now known under the name of “quantum probability”. At least some of the fundamental motivations for undertaking such investigations can be compared with the two basic questions of Frisch and Bourret:

(I) Non-commutative probability theory is meant as a generalization of classical probability theory to the description of quantum systems. Thus first of all their objects are operators on some Hilbert spaces having a meaning as non-commutative analogues of the probabilistic notions of random variables, stochastic processes, etc.

(II) In many investigations in this area one also tries to establish connections between non-commutative and classical concepts. The aim of this is twofold. On one side, one hopes to get a better understanding of classical problems by embedding them into a bigger non-commutative context. Thus, e.g., the Azéma martingale, although classically not distinguished within the class of all martingales, behaves in some respects like a Brownian motion [Par1]. The non-commutative “explanation” for this fact comes from the observation of Schürmann [Sch] that this martingale is one component of a non-commutative process with independent increments. In the other direction, one hopes to get a classical picture (featuring trajectories) of some aspects of quantum problems. Of course, a total reduction to classical concepts is in general not possible, but partial aspects may sometimes allow a classical interpretation.

It was in this context of quantum probability where two of the present authors [BSp1] reintroduced the $q$-relations – without knowing of, but much in the same spirit as [FB]. Around the same time the $q$-relations were also proposed by Greenberg [Gre] as an example for particles with “infinite statistics”.

The main progress in connection with this renewed interest was the solution of the realization problem of Frisch and Bourret. There exist now different proofs for the existence of the Fock representation of the $q$-relations for all $q$ with $-1 \leq q \leq 1$ [BSp1, Zag, Fiv, Spe1, BSp3, YW].

In [NSp], the idea of Frisch and Bourret to use the $q$-relations as a model for a generalized noise was pursued further and the Greens function for such dynamical problems could be calculated for one special choice of the covariance function – namely for the case of the exponential covariance. We will call this special $q$-process in the following $q$-Ornstein-Uhlenbeck process. It soon became clear that the special status of the exponential covariance is connected with some kind of (non-commutative) Markovianity – as we will see the $q$-Ornstein-Uhlenbeck process is the only stationary $q$-Gaussian Markov process. But using the general theory of Kümmerer on non-commutative stationary Markov processes [Kum1, Kum2] this readily implies the existence of a classical version (being itself a classical Markov process) of the $q$-Ornstein-Uhlenbeck process. Thus we got a positive solution of the random representation problem of Frisch and Bourret in this case. However, the status of the other $q$-Gaussian processes, in particular $q$-Brownian motion, remained unclear.

Motivated by our preliminary results, Biane [Bia1] (see also [Bia2, Bia3]) undertook a deep and beautiful analysis of the free ($q$ = 0) case and showed the remarkable result that all processes with free increments are Markovian and thus possess classical versions (with a quite explicit calculation rule for the corresponding transition probabilities). This includes in particular the case of free Brownian motion.

Inspired by this work we could extend our investigations from the case of the $q$-Ornstein-Uhlenbeck process to all $q$-Gaussian processes. The results are presented in this paper.

Up to now there is only one strategy for establishing the existence of a classical version of a non-commutative process, namely by showing that the process is Markovian. That this implies the existence of a classical version follows by general arguments, the main point is to show that we have this property in the concrete case. Whereas Biane could use the quite developed theory of freeness [VDN] to prove Markovianity for processes with free increments, there is at the moment (and probably also in the future [Spe2]) no kind of $q$-freeness for general $q$. Thus another feature of our considered class of processes is needed to attack the problem of Markovianity. It is the aim of this paper to convince the reader of the fact that the $q$-analogue of Gaussianity will do this job.

The essential idea of Gaussianity is that one can pull back all considerations from the measure theoretic (or, in the non-commutative frame, from the operator algebraic) level to an underlying Hilbert space, thus in the end one essentially has to deal with linear problems. The main point is that this transcription between the linear and the algebraic level exists in a consistent way. The best way to see and describe this is by presenting a functor (“second quantization”) which translates the Hilbert space properties into operator algebraic properties. Our basic considerations will therefore be on the existence and nice properties of the $q$-analogue of this functor. Having this functor, the rest is mainly linear theory on the Hilbert space level. It turns out that all relevant questions on our $q$-Gaussian processes can be characterized totally in terms of the corresponding covariance function. In particular, it becomes quite easy to decide whether such a process is Markovian or not.

The paper is organized as follows. In Sect. 1 we recall some basic facts about the $q$-Fock space and its relevant operators. Furthermore we collect in this section the needed combinatorial results, in particular on $q$-Hermite polynomials. Section 2 is devoted to the presentation of the functor $\Gamma_q$ of second quantization. The main results (apart from the existence of this object) are the facts that the associated von Neumann algebras are in the infinite dimensional case non-injective $II_1$-factors and that the functor maps contractions into completely positive maps. Having this $q$-Gaussian functor the definition and investigation of properties of $q$-Gaussian processes (like Markovianity or martingale property) is quite canonical and parallels the classical case. Thus our presentation of these aspects, in Sect. 3, will be quite condensed. Sect. 4 contains the classical interpretation of the $q$-Gaussian Markov processes. As pointed out above general arguments ensure the existence of classical versions for these processes. But we will see that we can also derive quite concrete formulas for the corresponding transition probabilities.

## 1. Preliminaries on the $q$-Fock Space

Let $q \in (-1, 1)$ be fixed in the following.

For a complex Hilbert space $\mathcal{H}$ we define its $q$-Fock space $\mathcal{F}_q(\mathcal{H})$ as follows: Let $\mathcal{F}^{finite}(\mathcal{H})$ be the linar span of vectors of the form $f_1 \otimes \ldots \otimes f_n \in \mathcal{H}^{\otimes n}$ (with varying $n \in \mathbb{N}_0$), where we put $\mathcal{H}^{\otimes 0} \cong \mathbb{C}\,\Omega$ for some distinguished vector $\Omega$, called vacuum. On $\mathcal{F}^{finite}(\mathcal{H})$ we consider the sesquilinear form $\langle \cdot, \cdot \rangle_q$ given by a sesquilinear extension of

$$\langle f_1 \otimes \ldots \otimes f_n, g_1 \otimes \ldots \otimes g_m \rangle_q := \delta_{nm} \sum_{\pi \in S_n} q^{i(\pi)} \langle f_1, g_{\pi(1)} \rangle \ldots \langle f_n, g_{\pi(n)} \rangle,$$

where $S_n$ denotes the symmetric group of permutations of $n$ elements and $i(\pi)$ is the number of inversions of the permutation $\pi \in S_n$ defined by

$$i(\pi) := \#\{(i,j) \mid 1 \leq i < j \leq n, \pi(i) > \pi(j)\}.$$

Another way to describe $\langle \cdot, \cdot \rangle_q$ is by introducing the operator $P_q$ on $\mathcal{F}^{finite}(\mathcal{H})$ by a linear extension of

$$\begin{aligned} P_q \,\Omega &= \Omega, \\ P_q f_1 \otimes \ldots \otimes f_n &= \textstyle\sum_{\pi \in S_n} q^{i(\pi)} f_{\pi(1)} \otimes \ldots \otimes f_{\pi(n)}. \end{aligned}$$

Then we can write

$$\langle \xi, \eta \rangle_q = \langle \xi, P_q \eta \rangle_0 \qquad (\xi, \eta \in \mathcal{F}^{finite}(\mathcal{H})),$$

where $\langle \cdot, \cdot \rangle_0$ is the scalar product on the usual full Fock space

$$\mathcal{F}_0(\mathcal{H}) = \bigoplus_{n \geq 0} \mathcal{H}^{\otimes n}.$$

One of the main results of [BSp1] (see also [BSp3, Fiv, Spe1, Zag]) was the strict positivity of $P_q$, i.e. $\langle \xi, \xi \rangle_q > 0$ for $0 \neq \xi \in \mathcal{F}^{finite}(\mathcal{H})$. This allows the following definitions.

**Definition 1.1.** *1) The $q$-Fock space $\mathcal{F}_q(\mathcal{H})$ is the completion of $\mathcal{F}^{finite}(\mathcal{H})$ with respect to $\langle\cdot,\cdot\rangle_q$.*
*2) Given $f \in \mathcal{H}$, we define the creation operator $a^*(f)$ and the annihilation operator $a(f)$ on $\mathcal{F}_q(\mathcal{H})$ by*

$$\begin{aligned} a^*(f)\,\Omega &= f, \\ a^*(f)f_1 \otimes \ldots \otimes f_n &= f \otimes f_1 \otimes \ldots \otimes f_n, \end{aligned}$$

*and*

$$\begin{aligned} a(f)\,\Omega &= 0, \\ a(f)f_1 \otimes \ldots \otimes f_n &= \sum_{i=1}^n q^{i-1}\langle f, f_i\rangle f_1 \otimes \ldots \otimes \check{f}_i \otimes \ldots \otimes f_n, \end{aligned}$$

*where the symbol $\check{f}_i$ means that $f_i$ has to be deleted in the tensor.*

*Remark 1.2.* The operators $a(f)$ and $a^*(f)$ are bounded operators on $\mathcal{F}_q(\mathcal{H})$ with

$$\|a(f)\|_q = \|a^*(f)\|_q = \begin{cases} \|f\|/\sqrt{1-q}, & 0 \le q < 1 \\ \|f\|, & -1 < q \le 0, \end{cases}$$

and they are adjoints of each other with respect to our scalar product $\langle\cdot,\cdot\rangle_q$. Furthermore, they fulfill the $q$-relations

$$a(f)a^*(g) - qa^*(g)a(f) = \langle f, g\rangle \cdot \mathbf{1} \qquad (f, g \in \mathcal{H}).$$

*Notation 1.3.* For a linear operator $T : \mathcal{H} \to \mathcal{H}'$ between two complex Hilbert spaces we denote by $\mathcal{F}(T) : \mathcal{F}^{finite}(\mathcal{H}) \to \mathcal{F}^{finite}(\mathcal{H}')$ the linear extension of

$$\begin{aligned} \mathcal{F}(T)\,\Omega &= \Omega, \\ \mathcal{F}(T)f_1 \otimes \ldots \otimes f_n &= (Tf_1) \otimes \ldots \otimes (Tf_n). \end{aligned}$$

In order to keep the notation simple we denote the vacuum for $\mathcal{H}$ and the vacuum for $\mathcal{H}'$ by the same symbol $\Omega$.

It is clear that $\mathcal{F}(T)$ can be extended to a bounded operator $\mathcal{F}_0(T) : \mathcal{F}_0(\mathcal{H}) \to \mathcal{F}_0(\mathcal{H}')$ exactly if $T$ is a contraction, i.e. if $\|T\| \le 1$. The following lemma ensures that the same is true for all other $q \in (-1, 1)$, too.

**Lemma 1.4.** *Let $\mathcal{T} : \mathcal{F}^{finite}(\mathcal{H}) \to \mathcal{F}^{finite}(\mathcal{H}')$ be a linear operator which fulfills $P'_q\mathcal{T} = \mathcal{T}P_q$, where $P_q$ and $P'_q$ are the operators on $\mathcal{F}^{finite}(\mathcal{H})$ and $\mathcal{F}^{finite}(\mathcal{H}')$, respectively, which define the respective scalar product $\langle\cdot,\cdot\rangle_q$. Then one has $\|\mathcal{T}\|_q = \|\mathcal{T}\|_0$. Hence, if $\|\mathcal{T}\|_0 < \infty$, then $\mathcal{T}$ can, for each $q \in (-1, 1)$, be extended to a bounded operator from $\mathcal{F}_q(\mathcal{H})$ to $\mathcal{F}_q(\mathcal{H}')$.*

*Proof.* Let $\xi \in \mathcal{F}^{finite}(\mathcal{H})$. Then

$$\begin{aligned} \|\mathcal{T}\xi\|_q^2 &= \langle \mathcal{T}\xi, \mathcal{T}\xi\rangle_q \\ &= \langle \mathcal{T}\xi, P'_q\mathcal{T}\xi\rangle_0 \\ &= \langle P_q^{1/2}\xi, \mathcal{T}^*\mathcal{T}P_q^{1/2}\xi\rangle_0 \\ &\le \|\mathcal{T}^*\mathcal{T}\|_0\,\langle P_q^{1/2}\xi, P_q^{1/2}\xi\rangle_0 \\ &= \|\mathcal{T}^*\mathcal{T}\|_0\,\|\xi\|_q^2, \end{aligned}$$

which implies

$$\|\mathcal{T}\|_q^2 \le \|\mathcal{T}^*\mathcal{T}\|_0 \le \|\mathcal{T}^*\|_0\,\|\mathcal{T}\|_0 = \|\mathcal{T}\|_0^2,$$

and thus $\|\mathcal{T}\|_q \le \|\mathcal{T}\|_0$. Since we can estimate in the same way, by replacing $P_q$ by $P_q^{-1}$ and $P'_q$ by $P_q'^{-1}$, also $\|\mathcal{T}\|_0 \le \|\mathcal{T}\|_q$, we get the assertion. $\square$

*Notation 1.5.* For a contraction $T : \mathcal{H} \to \mathcal{H}'$, we denote the extension of $\mathcal{F}(T)$ from $\mathcal{F}^{finite}(\mathcal{H}) \to \mathcal{F}^{finite}(\mathcal{H}')$ to $\mathcal{F}_q(\mathcal{H}) \to \mathcal{F}_q(\mathcal{H}')$ by $\mathcal{F}_q(T)$.

*Remark 1.6.* 1) One might call $\mathcal{F}_q(T)$ the second quantization of $T$, but we will reserve this name for the restriction of $\mathcal{F}_q(T)$ to some operator algebra lying in $\mathcal{F}_q(\mathcal{H})$ – see the next section, where we will also prove some positivity properties of this restricted version.
2) The operator $\mathcal{F}_q(T)$ and its differential version (in particular the number operator) were also considered in [Wer] and [Sta, Mol], respectively.
3) It is clear that $\mathcal{F}_q(\cdot)$ behaves nicely with respect to composition and taking adjoints, i.e.

$$\mathcal{F}_q(\mathbf{1}) = \mathbf{1}, \qquad \mathcal{F}_q(ST) = \mathcal{F}_q(S)\mathcal{F}_q(T), \qquad \mathcal{F}_q(T^*) = \mathcal{F}_q(T)^*,$$

but not with respect to the additive structure, i.e.

$$\mathcal{F}_q(T + S) \neq \mathcal{F}_q(T) + \mathcal{F}_q(S) \qquad \text{in general.}$$

In the context of the $q$-relations one usually encounters some kind of $q$-combinatorics. Let us just recall the basic facts.

*Notation 1.7.* We put for $n \in \mathbb{N}_0$,

$$[n]_q := \frac{1 - q^n}{1 - q} = 1 + q + \ldots + q^{n-1} \qquad ([0]_q := 0).$$

Then we have the $q$-factorial

$$[n]_q! := [1]_q \ldots [n]_q, \qquad [0]_q! := 1,$$

and a $q$-binomial coefficient

$$\binom{n}{k}_q := \frac{[n]_q!}{[k]_q![n-k]_q!} = \prod_{i=1}^{n-k} \frac{1 - q^{k+i}}{1 - q^i}.$$

Another quite frequently used symbol is the $q$-analogue of the Pochhammer symbol

$$(a;q)_n := \prod_{j=0}^{n-1}(1 - aq^j) \qquad \text{in particular} \qquad (a;q)_\infty := \prod_{j=0}^{\infty}(1 - aq^j).$$

The importance of these concepts in connection with the $q$-relations can be seen from the following $q$-binomial theorem, which is by now quite standard.

**Proposition 1.8.** *Let $x$ and $y$ be indeterminates which $q$-commute in the sense $xy = qyx$. Then one has for $n \in \mathbb{N}$,*

$$(x + y)^n = \sum_{k=0}^{n} \binom{n}{k}_q y^k x^{n-k}.$$

*Proof.* This is just induction and the easily checked equality

$$\binom{n}{k}_q + q^k \binom{n}{k+1}_q = \binom{n+1}{k+1}_q. \qquad \square$$

In the same way as the usual Hermite polynomials are connected to the bosonic relations, the $q$-relations are linked to $q$-analogues of the Hermite polynomials.

**Definition 1.9.** *The polynomials $H_n^{(q)}$ ($n \in \mathbb{N}_0$), determined by*

$$H_0^{(q)}(x) = 1, \qquad H_1^{(q)}(x) = x,$$

*and*

$$xH_n^{(q)}(x) = H_{n+1}^{(q)}(x) + [n]_q H_{n-1}^{(q)}(x) \qquad (n \geq 1)$$

*are called **$q$-Hermite polynomials**.*

We recall two basic facts about these polynomials which will be fundamental for our investigations on the classical aspects of $q$-Gaussian processes.

**Theorem 1.10.** *1) Let $\nu_q$ be the measure on the interval $[-2/\sqrt{1-q}, 2/\sqrt{1-q}]$ given by*

$$\nu_q(dx) = \frac{1}{\pi}\sqrt{1-q}\sin\theta \prod_{n=1}^{\infty}(1-q^n)|1-q^n e^{2i\theta}|^2 dx,$$

*where*

$$x = \frac{2}{\sqrt{1-q}}\cos\theta \qquad \text{with } \theta \in [0,\pi].$$

*Then the $q$-Hermite polynomials are orthogonal with respect to $\nu_q$, i.e.*

$$\int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} H_n(x)H_m(x)\nu_q(dx) = \delta_{nm}[n]_q!.$$

*2) Let $r > 0$ and $x, y \in [-2/\sqrt{1-q}, 2/\sqrt{1-q}]$. Denote by $p_r^{(q)}(x,y)$ the kernel*

$$p_r^{(q)}(x,y) := \sum_{n=0}^{\infty} \frac{r^n}{[n]_q!} H_n^{(q)}(x)H_n^{(q)}(y).$$

*Then we have with*

$$x = \frac{2}{\sqrt{1-q}}\cos\varphi, \qquad y = \frac{2}{\sqrt{1-q}}\cos\psi$$

*the formula*

$$p_r^{(q)}(x,y) = \frac{(r^2;q)_\infty}{|(re^{i(\varphi+\psi)};q)_\infty(re^{i(\varphi-\psi)};q)_\infty|^2}.$$

*In particular, for $q = 0$, we get*

$$p_r^{(0)}(x,y) = \frac{1-r^2}{(1-r^2)^2 - r(1+r^2)xy + r^2(x^2+y^2)}.$$

As usual in $q$-mathematics these formulas are quite old, namely the orthogonalizing measure $\nu_q$ was calculated by Szego [Sze], whereas the kernel $p_r^{(q)}(x,y)$ goes even back to Rogers [Rog]. For more recent treatments, see [Bre, ISV, GR], in connection with the $q$-Fock space also [LM1, LM2].

## 2. Second Quantization – The Functor $\Gamma_q$

An abstract way of dealing with classical Gaussian processes is by using the Gaussian functor $\Gamma$. This is a functor from real Hilbert spaces and contractions to commutative von Neumann algebras with specified trace-state and unital trace preserving completely positive maps [Nel1, Nel2, Gro, Sim1, Sim2]. Essentially, this point of view can be traced back to Segal [Seg]. Fermionic and free analogues of this functor are also known, see, e.g., [Wil, CL, Voi, VDN].

In this section we will present a $q$-analogue of the Gaussian functor. Namely, to each real Hilbert space, $\mathcal{H}$, we will associate a von Neumann algebra with specified trace-state, $(\Gamma_q(\mathcal{H}), \mathrm{E})$, and to every contraction $T : \mathcal{H} \to \mathcal{H}'$ a unital completely positive trace preserving map $\Gamma_q(T) : \Gamma_q(\mathcal{H}) \to \Gamma_q(\mathcal{H}')$.

**Definition 2.1.** *Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{H}_\mathbb{C}$ its complexification $\mathcal{H}_\mathbb{C} = \mathcal{H} \oplus i\mathcal{H}$. Put, for $f \in \mathcal{H}$,*

$$\omega(f) := a(f) + a^*(f) \in B(\mathcal{F}_q(\mathcal{H}_\mathbb{C}))$$

*and denote by $\Gamma_q(\mathcal{H}) \subset B(\mathcal{F}_q(\mathcal{H}_\mathbb{C}))$ the von Neumann algebra generated by all $\omega(f)$,*

$$\Gamma_q(\mathcal{H}) := \mathrm{vN}(a(f) + a^*(f) \mid f \in \mathcal{H}).$$

*Notation 2.2.* We denote by

$$\mathrm{E} : \Gamma_q(\mathcal{H}) \to \mathbb{C}$$

the vacuum expectation state on $\Gamma_q(\mathcal{H})$ given by

$$\mathrm{E}[X] := \langle \Omega, X\,\Omega \rangle_q \qquad (X \in \Gamma_q(\mathcal{H})).$$

We recall some basic facts about $\Gamma_q(\mathcal{H})$ in the following proposition.

**Proposition 2.3.** *The vacuum $\Omega$ is a cyclic and separating trace-vector for $\Gamma_q(\mathcal{H})$, hence the vacuum expectation* E *is a faithful normal trace on $\Gamma_q(\mathcal{H})$ and $\Gamma_q(\mathcal{H})$ is a finite von Neumann algebra in standard form.*

*Proof.* See Theorems 4.3 and 4.4 in [BSp3]. □

The first part of the proposition yields in particular that the mapping

$$\begin{aligned} \Gamma_q(\mathcal{H}) &\to \mathcal{F}_q(\mathcal{H}_\mathbb{C}), \\ X &\mapsto X\,\Omega \end{aligned}$$

is injective, in this way we can identify each $X \in \Gamma_q(\mathcal{H})$ with some element of the $q$-Fock space $\mathcal{F}_q(\mathcal{H}_\mathbb{C})$.

*Notation 2.4.* 1) Let us denote by

$$L_q^\infty(\mathcal{H}) := \Gamma_q(\mathcal{H})\,\Omega$$

the image of $\Gamma_q(\mathcal{H})$ under the mapping $X \mapsto X\,\Omega$.
2) We also put

$$L_q^2(\mathcal{H}) := \mathcal{F}_q(\mathcal{H}_\mathbb{C}).$$

**Definition 2.5.** *Let $\Psi : L_q^\infty(\mathcal{H}) \to \Gamma_q(\mathcal{H})$ be the identification of $L_q^\infty(\mathcal{H})$ with $\Gamma_q(\mathcal{H})$ given by the requirement*

$$\Psi(\xi)\,\Omega = \xi \qquad \text{for} \quad \xi \in L_q^\infty(\mathcal{H}) \subset L_q^2(\mathcal{H}) = \mathcal{F}_q(\mathcal{H}_\mathbb{C}).$$

*Remark 2.6.* 1) Of course, not each element of the $q$-Fock space comes from an $X \in \Gamma_q(\mathcal{H})$, but the main relation for observing the cyclicity of $\Omega$, namely

$$f_1 \otimes \ldots \otimes f_n = \omega(f_1) \ldots \omega(f_n)\, \Omega - \eta \qquad \text{with} \quad \eta \in \bigoplus_{l=0}^{n-1} \mathcal{H}^{\otimes l},$$

yields that we have at least $f_1 \otimes \ldots \otimes f_n \in L_q^\infty(\mathcal{H})$.
2) In a quantum field theoretic context [Sim1, Sim2] the operator $\Psi(f_1 \otimes \ldots \otimes f_n)$ would be called "Wick product" and denoted by

$$\Psi(f_1 \otimes \ldots \otimes f_n) =: \omega(f_1) \ldots \omega(f_n) : .$$

3) In a quantum probabilistic context [Par2, Mey] $\Psi$ would correspond to taking an iterated quantum stochastic integral: For $\mathcal{H}_{\mathbb{C}} = L^2(\mathbb{R})$ and $\xi = f_1 \otimes \ldots \otimes f_n$ with $\xi(t_1, \ldots, t_n) = f_1(t_1) \ldots f_n(t_n)$ one would denote

$$\Psi(\xi) = \int \xi(t_1, \ldots, t_n) d\omega(t_1) \ldots d\omega(t_n)$$

and call $\xi$ the "Maassen kernel" of $\Psi(\xi)$.

The explicit form of our Wick products is given in the following proposition.

**Proposition 2.7.** *We have for $n \in \mathbb{N}$ and $f_1, \ldots, f_n \in \mathcal{H}$ the normal ordered representation*

$$\Psi(f_1 \otimes \ldots \otimes f_n) = \\ = \sum_{\substack{k,l=0,\ldots,n \\ k+l=n}} \sum_{\substack{I_1=\{i(1),\ldots,i(k)\} \\ I_2=\{j(1),\ldots,j(l)\} \\ \text{with} \\ I_1 \cup I_2 = \{1,\ldots,n\} \\ I_1 \cap I_2 = \emptyset}} a^*(f_{i(1)}) \ldots a^*(f_{i(k)}) a(f_{j(1)}) \ldots a(f_{j(l)}) \cdot q^{i(I_1,I_2)},$$

*where*

$$i(I_1, I_2) := \#\{(p,q) \mid 1 \le p \le k, 1 \le q \le l, i(p) > j(q)\}.$$

Denote by $X$ the right-hand side of the above relation. It is clear that $X\,\Omega = f_1 \otimes \ldots \otimes f_n$, the problem is to see that $X$ can be expressed in terms of the $\omega$'s.

*Proof.* Note that the formula is true for

$$\Psi(f) = \omega(f) = a(f) + a^*(f)$$

and that the definition of $a^*(f)$ and of $a(f)$ gives

$$\Psi(f \otimes f_1 \otimes \ldots \otimes f_n) = \omega(f)\, \Psi(f_1 \otimes \ldots \otimes f_n) - \sum_{i=1}^{n} q^{i-1} \langle f, f_i \rangle\, \Psi(f_1 \otimes \ldots \otimes \check{f}_i \otimes \ldots \otimes f_n).$$

From this the assertion follows by induction. □

Note that $\Psi(f_1 \otimes \ldots \otimes f_n)$ is just given by multiplying out $\omega(f_1) \ldots \omega(f_n)$ and bringing all appearing terms with the help of the relation $aa^* = qa^*a$ into a normal ordered form – i.e. we throw away all normal ordered terms in $\omega(f_1) \ldots \omega(f_n)$ which have less than $n$ factors. Thus, for the special case $f_1 = \ldots = f_n$, we are in the realm of the $q$-binomial theorem and we have the following nice formula.

**Corollary 2.8.** *We have for $n \in \mathbb{N}$ and $f \in \mathcal{H}$,*

$$\Psi(f^{\otimes n}) = \sum_{k=0}^{n} \binom{n}{k}_q a^*(f)^k a(f)^{n-k}.$$

Instead of writing $\Psi(f^{\otimes n})$ in a normal ordered form we can also express it in terms of $\omega(f)$ with the help of the $q$-Hermite polynomials.

**Proposition 2.9.** *We have for $n \in \mathbb{N}_0$ and $f \in \mathcal{H}$ with $\|f\| = 1$ the representation*

$$\Psi(f^{\otimes n}) = H_n^{(q)}(\omega(f)).$$

*Proof.* This follows by the fact that the $\Psi(f^{\otimes n})$ fulfill the same recurrence relation as the $H_n^{(q)}(\omega(f))$, namely

$$\omega(f)\,\Psi(f^{\otimes n}) = \Psi(f^{\otimes(n+1)}) + [n]_q\,\Psi(f^{\otimes(n-1)})$$

and that we have the same initial conditions

$$\Psi(f^{\otimes 0}) = \mathbf{1}, \qquad \Psi(f^{\otimes 1}) = \omega(f). \qquad \square$$

We know [Voi, VDN] that for $q = 0$ the von Neumann algebra $\Gamma_0(\mathcal{H})$ is isomorphic to the von Neumann algebra of the free group on $\dim \mathcal{H}$ generators – in particular, it is a non-injective $\mathrm{II}_1$-factor for $\dim \mathcal{H} \geq 2$. We conjecture non-injectivity and factoriality in the case $\dim \mathcal{H} \geq 2$ for arbitrary $q \in (-1, 1)$, but up to now we can only show the following.

**Theorem 2.10.** *1) For $-1 < q < 1$ and $\dim \mathcal{H} > 16/(1 - |q|)^2$ the von Neumann algebra $\Gamma_q(\mathcal{H})$ is not injective.*
*2) If $-1 < q < 1$ and $\dim \mathcal{H} = \infty$ then $\Gamma_q(\mathcal{H})$ is a $II_1$-factor.*

*Proof.* 1) This was shown in a more general context in Theorem 4.2 in [BSp3].
2) Let $\{e_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $\mathcal{H}$. Fix $n \in \mathbb{N}_0$ and $r(1), \dots, r(n) \in \mathbb{N}$ and consider the operator

$$X := \Psi(e_{r(1)} \otimes \dots \otimes e_{r(n)}).$$

(For $n = 0$ this will be understood as $X = \mathbf{1}$.) We put

$$\phi_m(X) := \frac{1}{m} \sum_{i=1}^{m} \omega(e_i) X \omega(e_i) \qquad (m \in \mathbb{N})$$

and claim that $\phi_m(X)$ converges for $m \to \infty$ weakly to $\phi(X) := q^n X$. Because of the $m$-independent estimate

$$\|\phi_m(X)\|_q \leq \|X\|_q \, \|\omega(e_1)\|_q^2$$

it suffices to show

$$\lim_{m\to\infty} \langle \xi, \phi_m(X)\eta \rangle_q = \langle \xi, \phi(X)\eta \rangle_q$$

for all $\xi, \eta \in \mathcal{F}_q(\mathcal{H}_\mathbb{C})$ of the form

$$\xi = e_{a(1)} \otimes \dots \otimes e_{a(u)}, \qquad \eta = e_{b(1)} \otimes \dots \otimes e_{b(v)}$$

with $u, v \in \mathbb{N}_0$, $a(1), \dots, a(u), b(1), \dots, b(v) \in \mathbb{N}$ (for $u = 0$ we put $\xi = \Omega$). To see this, put

$$m_0 := \max\{a(1), \dots, a(u), b(1), \dots, b(v), r(1), \dots, r(n)\}.$$

Since $|\langle \xi, \omega(e_i) X \omega(e_i) \eta \rangle_q| \leq M$ for some $M$ (independent of $i$), we have

$$\begin{aligned} \lim_{n\to\infty} \langle \xi, \phi_m(X)\eta \rangle_q &= \lim_{n\to\infty} \tfrac{1}{m} \textstyle\sum_{i=m_0+1}^m \langle \xi, \omega(e_i) X \omega(e_i) \eta \rangle_q \\ &= \lim_{n\to\infty} \tfrac{1}{m} \textstyle\sum_{i=m_0+1}^m \langle \xi, a(e_i)\, \Psi(e_{r(1)} \otimes \dots \otimes e_{r(n)}) a^*(e_i) \eta \rangle_q. \end{aligned}$$

By Prop. 2.7, $\Psi(e_{r(1)} \otimes \dots \otimes e_{r(n)})$ is now a linear combination of terms of the form $Y = Y_1 Y_2$ with

$$Y_1 = a^*(e_{r(i(1))}) \dots a^*(e_{r(i(k))}) \qquad \text{and} \qquad Y_2 = a(e_{r(j(1))}) \dots a(e_{r(j(l))})$$

with $k + l = n$. Each such term gives, for $i > m_0$, a contribution

$$\begin{aligned} \langle \xi, a(e_i) Y a^*(e_i) \eta \rangle_q &= \langle \xi, a(e_i) Y_1 Y_2 a^*(e_i) \eta \rangle_q \\ &= q^{k+l} \langle \xi, Y_1 a(e_i) a^*(e_i) Y_2 \eta \rangle_q \\ &= q^n \langle \xi, Y_1 (\mathbf{1} + q a^*(e_i) a(e_i)) Y_2 \eta \rangle_q \\ &= q^n \langle \xi, Y_1 Y_2 \eta \rangle_q \\ &= q^n \langle \xi, Y \eta \rangle_q, \end{aligned}$$

and hence

$$\begin{aligned} \lim_{m\to\infty} \langle \xi, \phi_m(X)\eta \rangle_q &= \lim_{m\to\infty} \tfrac{1}{m} \textstyle\sum_{i=m_0+1}^m q^n \langle \xi,\ \Psi(e_{r(1)} \otimes \dots \otimes e_{r(n)}) \eta \rangle_q \\ &= \langle \xi, q^n X \eta \rangle_q. \end{aligned}$$

Thus we have shown

$$\operatorname*{w-lim}_{m\to\infty} \phi_m(X) = \phi(X).$$

Let now tr be a normalized normal trace on $\Gamma_q(\mathcal{H})$. Then

$$\begin{aligned} \mathrm{tr}[\phi(X)] &= \lim_{m\to\infty} \mathrm{tr}[\phi_m(X)] \\ &= \lim_{m\to\infty} \tfrac{1}{m} \textstyle\sum_{i=1}^m \mathrm{tr}[\omega(e_i) X \omega(e_i)] \\ &= \lim_{m\to\infty} \tfrac{1}{m} \textstyle\sum_{i=1}^m \mathrm{tr}[X \omega(e_i) \omega(e_i)] \\ &= \mathrm{tr}[X \cdot \lim_{m\to\infty} \tfrac{1}{m} \textstyle\sum_{i=1}^m \omega(e_i)\omega(e_i)] \\ &= \mathrm{tr}[X \phi(\mathbf{1})] \\ &= \mathrm{tr}[X]. \end{aligned}$$

Since $\phi^k(X) = q^{kn} X$ converges, for $k \to \infty$, (even in norm) to

$$\mathrm{E}[X] \cdot \mathbf{1} = \begin{cases} 0, & n \geq 1 \\ X = \mathbf{1}, & n = 0 \end{cases},$$

we obtain

$$\mathrm{tr}[X] = \lim_{k\to\infty} \mathrm{tr}[\phi^k(X)] = \mathrm{tr}[\lim_{k\to\infty} \phi^k(X)] = \mathrm{E}[X]\,\mathrm{tr}[\mathbf{1}] = \mathrm{E}[X].$$

Thus tr coincides on all operators of the form

$$X = \Psi(e_{r(1)} \otimes \dots \otimes e_{r(n)}) \qquad (n \in \mathbb{N}_0,\ r(1), \dots, r(n) \in \mathbb{N})$$

with our canonical trace E. Since the set of finite linear combinations of such operators $X$ is weakly dense in $\Gamma_q(\mathcal{H})$, we get the uniqueness of a normalized normal trace on $\Gamma_q(\mathcal{H})$, which implies that $\Gamma_q(\mathcal{H})$ is a factor. □

The second part of our $q$-Gaussian functor $\Gamma_q$ assigns to each contraction $T : \mathcal{H} \to \mathcal{H}'$ a map $\Gamma_q(T) : \Gamma_q(\mathcal{H}) \to \Gamma_q(\mathcal{H}')$. The idea is to extend $\Gamma_q(T)\omega(f) = \omega(Tf)$ in a canonical way to all of $\Gamma_q(\mathcal{H})$. In general, the $q$-relations prohibit the extension as a homomorphism, i.e.

$$\Gamma_q(T)\omega(f_1)\ldots\omega(f_n) \neq \omega(Tf_1)\ldots\omega(Tf_n) \qquad \text{in general.}$$

But what can be done is to demand the above relation for the normal ordered form, i.e.

$$\Gamma_q(T)\,\Psi(f_1 \otimes \ldots \otimes f_n) = \Psi(Tf_1 \otimes \ldots \otimes Tf_n) = \Psi(\mathcal{F}_q(T)f_1 \otimes \ldots \otimes f_n),$$

or

$$(\Gamma_q(T)X)\,\Omega = \mathcal{F}_q(T)(X\,\Omega).$$

Thus our second quantization $\Gamma_q(T)$ is the restriction of $\mathcal{F}_q(T)$ from $\mathcal{F}_q(\mathcal{H}) = L^2_q(\mathcal{H})$ to $\Gamma_q(\mathcal{H}) \cong L^\infty_q(\mathcal{H})$ and the question on the existence of $\Gamma_q(T)$ amounts to the problem whether $\mathcal{F}_q(T)(L^\infty_q(\mathcal{H})) \subset L^\infty_q(\mathcal{H}')$. We know that $\mathcal{F}_q(T)$ can be defined for $T$ a contraction and we will see in the next theorem that no extra condition is needed to ensure its nice behaviour with respect to $L^\infty_q$. The case $q = 0$ is due to Voiculescu [Voi, VDN].

**Theorem 2.11.** *1) Let $T : \mathcal{H} \to \mathcal{H}'$ be a contraction between real Hilbert spaces. There exists a unique map $\Gamma_q(T) : \Gamma_q(\mathcal{H}) \to \Gamma_q(\mathcal{H}')$ such that*

$$(\Gamma_q(T)X)\,\Omega = \mathcal{F}_q(T)(X\,\Omega).$$

*The map $\Gamma_q(T)$ is linear, bounded, completely positive, unital and preserves the canonical trace* E.
*2) If $T$ is isometric, then $\Gamma_q(T)$ is a faithful homomorphism, and if $T$ is the orthogonal projection onto a subspace, then $\Gamma_q(T)$ is a conditional expectation.*

*Proof.* Uniqueness of $\Gamma_q(T)$ follows from the fact that $\Omega$ is separating for $\Gamma_q(\mathcal{H}')$. To prove the existence and the properties of $\Gamma_q(T)$ we notice that any contraction $T$ can be factored [Hal] as $T = POI$ where
– $I : \mathcal{H} \to \mathcal{K} = \mathcal{H} \oplus \mathcal{H}'$ is an isometric embedding
– $O : \mathcal{K} \to \mathcal{K}$ is orthogonal
– $P : \mathcal{K} = \mathcal{H} \oplus \mathcal{H}' \to \mathcal{H}'$ is an orthogonal projection onto a subspace.
Thus if we prove our assertions for each of these three cases then we will also get the general statement for $\Gamma_q(T) = \Gamma_q(P)\,\Gamma_q(O)\,\Gamma_q(I)$.
a) Let $I : \mathcal{H} \to \mathcal{K} = \mathcal{H} \oplus \mathcal{H}'$ be an isometric embedding and $Q : \mathcal{K} \to \mathcal{K}$ the orthogonal projection onto $\mathcal{H}$. Then $\mathcal{F}_q(Q)$ is a projection in $\mathcal{F}_q(\mathcal{K}_\mathbb{C})$ and $\mathcal{F}_q(\mathcal{H}_\mathbb{C})$ can be identified with $\mathcal{F}_q(Q)\mathcal{F}_q(\mathcal{K}_\mathbb{C})$. Let us denote by $\omega_\mathcal{K}(f)$ the sum of the creation and annihilation operator on $\mathcal{F}_q(\mathcal{K}_\mathbb{C})$. If we put

$$\Gamma^\mathcal{K}_q(\mathcal{H}) := \mathrm{vN}(\omega_\mathcal{K}(f) \mid f \in \mathcal{H}) \subset B(\mathcal{F}_q(\mathcal{K}_\mathbb{C})),$$

then

$$\Gamma^\mathcal{K}_q(\mathcal{H})\mathcal{F}_q(\mathcal{H}_\mathbb{C}) \subset \mathcal{F}_q(\mathcal{H}_\mathbb{C}),$$

and we have the canonical identification

$$\Gamma_q(\mathcal{H}) \cong \Gamma^\mathcal{K}_q(\mathcal{H})\mathcal{F}_q(Q),$$

which gives a homomorphism (and thus a completely positive)

$$\Gamma_q(I) : \Gamma_q(\mathcal{H}) \to \Gamma_q(\mathcal{K}).$$

Faithfulness is clear since $\mathcal{F}_q(Q)\,\Omega = \Omega$ and $\Omega$ separating. This yields also that the trace is preserved.
b) Let $P : \mathcal{K} = \mathcal{H} \oplus \mathcal{H}' \to \mathcal{H}'$ be an orthogonal projection, i.e. $PP^* = \mathbf{1}_{\mathcal{H}'}$, where $P^* : \mathcal{H}' \to \mathcal{K}$ is the canonical inclusion. Then

$$\Gamma_q(P)X := \mathcal{F}_q(P)X\mathcal{F}_q(P^*) \qquad (X \in \Gamma_q(\mathcal{K}))$$

gives the right operator, because we have for $k, l \in \mathbb{N}_0$ and $f_1, \dots, f_k, g_1, \dots, g_l \in \mathcal{K}$,

$$\begin{aligned}\mathcal{F}_q(P)a^*(f_1)\dots a^*(f_k)a(g_1)\dots a(g_l)\mathcal{F}_q(P^*) &= \\ &= a^*(Pf_1)\dots a^*(Pf_k)\mathcal{F}_q(P)\mathcal{F}_q(P^*)a(Pg_1)\dots a(Pg_l) \\ &= a^*(Pf_1)\dots a^*(Pf_k)a(Pg_1)\dots a(Pg_l).\end{aligned}$$

By its concrete form, $\Gamma_q(P)$ is a conditional expectation and

$$\mathrm{E}[\mathcal{F}_q(P)X\mathcal{F}_q(P^*)] = \langle \mathcal{F}_q(P^*)\,\Omega, X\mathcal{F}_q(P^*)\,\Omega\rangle_q = \langle\, \Omega, X\,\Omega\rangle_q = \mathrm{E}[X]$$

shows that it preserves the trace.
c) Let $O : \mathcal{K} \to \mathcal{K}$ be orthogonal, i.e. $OO^* = O^*O = \mathbf{1}_{\mathcal{K}}$. Then, as in b),

$$\Gamma_q(O)X = \mathcal{F}_q(O)X\mathcal{F}_q(O^*),$$

which is, by

$$\mathcal{F}_q(O^*)\mathcal{F}_q(O) = \mathcal{F}_q(\mathbf{1}_{\mathcal{K}}) = \mathbf{1}_{\mathcal{F}_q(\mathcal{K}_{\mathbb{C}})}$$

also a faithful homomorphism. □

Instead of working on the level of von Neumann algebras we could also consider the $C^*$-analogues of the above constructions. This would be quite similar. We just indicate the main points.

**Definition 2.12.** *Let $\mathcal{H}$ be a real Hilbert space and $\mathcal{H}_{\mathbb{C}}$ its complexification $\mathcal{H}_{\mathbb{C}} = \mathcal{H} \oplus i\mathcal{H}$. Put, for $f \in \mathcal{H}$,*

$$\omega(f) := a(f) + a^*(f) \in B(\mathcal{F}_q(\mathcal{H}_{\mathbb{C}})),$$

*and denote by $\Phi_q(\mathcal{H}) \subset B(\mathcal{F}_q(\mathcal{H}_{\mathbb{C}}))$ the $C^*$-algebra generated by all $\omega(f)$,*

$$\Phi_q(\mathcal{H}) := C^*(a(f) + a^*(f) \mid f \in \mathcal{H}).$$

Clearly, the vacuum is also a separating trace-vector for $\Phi_q(\mathcal{H})$ and, by Remark 2.6., it is also cyclic and $\Psi(f_1 \otimes \dots \otimes f_n) \in \Phi_q(\mathcal{H})$ for all $n \in \mathbb{N}_0$ and all $f_1, \dots, f_n \in \mathcal{H}$.

The most important fact for our later considerations is that $\Gamma_q(T)$ can also be restricted to the $C^*$-level.

**Theorem 2.13.** *1) Let $T : \mathcal{H} \to \mathcal{H}'$ be a contraction between real Hilbert spaces. There exists a unique map $\Phi_q(T) : \Phi_q(\mathcal{H}) \to \Phi_q(\mathcal{H}')$ such that*

$$(\Phi_q(T)X)\,\Omega = \mathcal{F}_q(T)(X\,\Omega).$$

*The map $\Phi_q(T)$ is linear, bounded, completely positive, unital and preserves the canonical trace* E.
*2) If $T$ is isometric, then $\Phi_q(T)$ is a faithful homomorphism, and if $T$ is the orthogonal projection onto a subspace, then $\Phi_q(T)$ is a conditional expectation.*
*3) We have $\Phi_q(T) = \Gamma_q(T)/\,\Phi_q(\mathcal{H})$.*

*Proof.* This is analogous to the proof of Theorem 2.11. □

We can now also prove the analogue of the second part of Theorem 2.10. The analogue of factoriality for $C^*$-algebras is simplicity.

**Theorem 2.14.** *If* $-1 < q < 1$ *and* $\dim \mathcal{H} = \infty$ *then* $\Phi_q(\mathcal{H})$ *is simple.*

*Proof.* Again, this is similar to the proof of the von Neumann algebra result. We just indicate the main steps.
We use the notations from the proof of Theorem 2.10. First, by norm estimates, one can show that the convergence $\lim_{m\to\infty} \phi_m(X) = \phi(X)$ for $X$ of the form $X := \Psi(e_{r(1)} \otimes \ldots \otimes e_{r(n)})$ is even a convergence in norm. Since $\phi(X)$ is nothing but $\phi(X) = \Gamma_q(q)X$, where $q$ is regarded as a multiplication operator on $\mathcal{H}$, we have, by 2.13, the bound

$$\|\phi(X)\|_q \leq \|X\|_q.$$

This together with the $m$-independent bound

$$\|\phi_m(X)\|_q \leq \|X\|_q \, \|\omega(e_1)\|_q^2$$

implies that

$$\lim_{m\to\infty} \phi_m(X) = \Gamma_q(q)X \qquad \text{uniformly for all } X \in \Phi_q(\mathcal{H}).$$

Now assume we have a non-trivial ideal $I$ in $\Phi_q(\mathcal{H})$ and consider a positive non-vanishing $X \in I$. Then $\phi_m(X) \in I$ for all $m \in \mathbb{N}$ and thus $\Gamma_q(q)X \in I$. Iterating shows $\Gamma_q(q^n)X \in I$ for all $n \in \mathbb{N}$ and because of the uniform convergence $\lim_{n\to\infty} \Gamma_q(q^n)X = \mathrm{E}[X]\mathbf{1}$, we obtain $\mathrm{E}[X]\mathbf{1} \in I$. The faithfulness of E implies then $I = \Phi_q(\mathcal{H})$. □

*Remark 2.15.* One might be tempted to conjecture that, for fixed $\mathcal{H}$, the von Neumann algebras $\Gamma_q(\mathcal{H})$ or the $C^*$-algebras $\Phi_q(\mathcal{H})$ are for all $q \in (-1, 1)$ isomorphic. At the moment, no results in this direction are known. One should note that there exist partial answers [JSW1, JSW2, JW, DN] to the analogous question for the $C^*$-algebra generated by $a(f), a^*(f)$ (not the sum) showing that at least for small values of $q$ and $n := \dim \mathcal{H}_\mathbb{C} < \infty$ these algebras are isomorphic to the ($q = 0$)-algebra, which is an extension of the Cuntz algebra $O_n$ by compact operators [Cun, Eva]. However, the methods used there do not extend to the case of $\Gamma_q(\mathcal{H})$ or $\Phi_q(\mathcal{H})$.

## 3. Non-commutative Aspects of $q$-Gaussian Processes

Before we define the notion of a $q$-Gaussian process, we want to present our general frame on non-commutative processes. By $T$ we will denote the range of our time parameter $t$, typically $T$ will be some interval in $\mathbb{R}$.

**Definition 3.1.** *1) Let $\mathcal{A}$ be a finite von Neumann algebra and $\varphi : \mathcal{A} \to \mathbb{C}$ a faithful normal trace on $\mathcal{A}$. Then we call the pair $(\mathcal{A}, \varphi)$ a* **(tracial) probability space***.*
*2) A* **random variable** *on $(\mathcal{A}, \varphi)$ is a self-adjoint operator $X \in \mathcal{A}$.*
*3) A* **stochastic process** *on $(\mathcal{A}, \varphi)$ is a family $(X_t)_{t\in T}$ of random variables $X_t \in \mathcal{A}$ $(t \in T)$.*
*4) The* **distribution** *of a random variable $X$ on $(\mathcal{A}, \varphi)$ is the probability measure $\nu$ on the spectrum of $X$ determined by*

$$\varphi(X^n) = \int x^n d\nu(x) \qquad \text{for all } n \in \mathbb{N}_0.$$

We should point out that there are also a lot of quantum probabilistic investigations in the context of more general, non-tracial situations, see e.g. [AFL, Kum1]. Of course, life becomes much harder there.

We will only consider centered Gaussian processes, thus a $q$-Gaussian process will be totally determined by its covariance. Since we would like to have realizations of our processes on *separable* Hilbert spaces, our admissible covariances are not just positive definite functions, but they should admit a separable representation.

**Definition 3.2.** *A function $c : T \times T \to \mathbb{R}$ is called a* **covariance function**, *if there exists a separable real Hilbert space $\mathcal{H}$ and vectors $f_t \in \mathcal{H}$ for all $t \in T$ such that*

$$c(s,t) = \langle f_s, f_t \rangle \qquad (s,t \in \mathcal{H}).$$

**Definition 3.3.** *Let $c : T \times T \to \mathbb{R}$ be a covariance function corresponding to a real Hilbert space $\mathcal{H}$ and vectors $f_t \in \mathcal{H}$ ($t \in T$). Then we put for all $t \in T$,*

$$X_t := \omega(f_t) \in \Gamma_q(\mathcal{H})$$

*and call the process $(X_t)_{t \in T}$ on $(\Gamma_q(\mathcal{H}), \mathrm{E})$ the* ***q*-Gaussian process with covariance *c***.

*Remark 3.4.* 1) Of course, the $q$-Gaussian process depends, up to isomorphism, only on $c$ and not on the special choice of $\mathcal{H}$ and $(f_t)_{t \in T}$.
2) We can characterize our $q$-Gaussian process also by the $q$-relations in the form

$$X_t = a_t + a_t^* \qquad \text{and} \qquad \mathrm{E}[\,\cdot\,] = \langle\, \Omega, \cdot\ \Omega \rangle,$$

where for all $s, t \in T$,

$$a_s a_t^* - q a_t^* a_s = c(s,t) \cdot \mathbf{1} \qquad \text{and} \qquad a_t\, \Omega = 0.$$

In this form our $q$-Gaussian processes were introduced by Frisch and Bourret [FB].

We can now define $q$-analogues of all classical Gaussian processes, just by choosing the appropriate covariance. In the following we consider three prominent examples.

**Definition 3.5.** *1) The $q$-Gaussian process $(X_t^{qBM})_{t \in [0,\infty)}$ with covariance*

$$c(s,t) = \min(s,t) \qquad (0 \le s,t < \infty)$$

*is called* ***q*-Brownian motion**.
*2) The $q$-Gaussian process $(X_t^{qBB})_{t \in [0,1]}$ with covariance*

$$c(s,t) = s(1-t) \qquad (0 \le s \le t \le 1)$$

*is called* ***q*-Brownian bridge**.
*3) The $q$-Gaussian process $(X_t^{qOU})_{t \in \mathbb{R}}$ with covariance*

$$c(s,t) = e^{-|t-s|} \qquad (s,t \in \mathbb{R})$$

*is called* ***q*-Ornstein-Uhlenbeck process**.

*Remark 3.6.* 1) That the three examples for $c$ are indeed covariance functions is clear by the existence of the respective classical processes, for a direct proof see, e.g., [Sim2].
2) The Ornstein-Uhlenbeck process is often also called an oscillator process, see [Sim2].

Let $(\mathcal{A}, \varphi)$ be a tracial probability space and let $\mathcal{B}$ be a von Neumann subalgebra of $\mathcal{A}$. Then we have (see, e.g., [Tak]) a unique conditional expectation ("partial trace") from $\mathcal{A}$ onto $\mathcal{B}$ which preserves the trace $\varphi$ – which we will denote in a probabilistic language by $\varphi[\,\cdot\,|\mathcal{B}]$. Thus in the frame of tracial probability spaces we always have the following canonical generalization of the classical Markov property (which says that the future depends on the past only through the present).

**Definition 3.7.** *Let $(\mathcal{A}, \varphi)$ be a probability space and $(X_t)_{t\in T}$ a stochastic process on $(\mathcal{A}, \varphi)$. Denote by*

$$\begin{aligned} \mathcal{A}_{t]} &:= \mathrm{vN}(X_u \mid u \le t) \subset \mathcal{A}, \\ \mathcal{A}_{[t]} &:= \mathrm{vN}(X_t) \qquad\quad \subset \mathcal{A}, \\ \mathcal{A}_{[t} &:= \mathrm{vN}(X_u \mid u \ge t) \subset \mathcal{A}. \end{aligned}$$

*We say that $(X_t)_{t\in T}$ is a* **Markov process** *if we have for all $s, t \in T$ with $s \le t$ the property*

$$\varphi[X|\mathcal{A}_{s]}] \in \mathcal{A}_{[s]} \qquad \text{for all } X \in \mathcal{A}_{[t]}.$$

Note that another canonical definition for the Markov property would be the requirement

$$\varphi[X|\mathcal{A}_{s]}] \in \mathcal{A}_{[s]} \qquad \text{for all } X \in \mathcal{A}_{[s}.$$

In the classical case this latter condition is equivalent to the one we use in Definition 3.7, but in the non-commutative case there is in general a difference. We have chosen the weaker condition, since this is sufficient to ensure the existence of transition operators (see Definition 4.3 and Theorem 4.4).

Now, the conditional expectations $\mathrm{E}[\,\cdot\,|\mathcal{A}_{s]}]$ in the case of $q$-Gaussian processes are quite easy to handle because they are nothing but the second quantization of projections in the underlying Hilbert space. Namely, consider a $q$-Gaussian process $(X_t)_{t\in T}$ corresponding to the real Hilbert space $\mathcal{H}$ and vectors $f_t$ ($t \in T$). Let us denote by

$$\begin{aligned} \mathcal{H}_{t]} &:= \mathrm{span}(f_u \mid u \le t) \subset \mathcal{H}, \\ \mathcal{H}_{[t]} &:= \mathbb{R} f_t \qquad\qquad\quad \subset \mathcal{H}, \\ \mathcal{H}_{[t} &:= \mathrm{span}(f_u \mid u \ge t) \subset \mathcal{H}, \end{aligned}$$

the respective Hilbert space analogues of $\mathcal{A}_{t]}$, $\mathcal{A}_{[t]}$, $\mathcal{A}_{[t}$. Then we have

$$\mathcal{A}_{t]} \cong \Gamma_q(\mathcal{H}_{t]}), \qquad \mathcal{A}_{[t]} \cong \Gamma_q(\mathcal{H}_{[t]}), \qquad \mathcal{A}_{[t} \cong \Gamma_q(\mathcal{H}_{[t}),$$

and $\mathrm{E}[\,\cdot\,|\mathcal{A}_{t]}] = \Gamma_q(P_{t]})$ is the second quantization of the orthogonal projection

$$P_{t]} : \mathcal{H} \to \mathcal{H}_{t]}.$$

Thus we can translate the Markov property for $q$-Gaussian processes into the following Hilbert space level statement.

**Proposition 3.8.** *Let $(X_t)_{t\in T}$ be a $q$-Gaussian process as above. It has the Markov property if and only if*

$$P_{s]}\mathcal{H}_{[t]} \subset \mathcal{H}_{[s]} \qquad \text{for all } s, t \in T \text{ with } s \le t.$$

Note that the stronger form of Markovianity, $\varphi[\mathcal{A}_{[s}, \mathcal{A}_{s]}] \subset \mathcal{A}_{[s]}$, corresponds for the $q$-Gaussian processes on the linear level to $P_{s]}\mathcal{H}_{[s} \subset \mathcal{H}_{[s]}$. But this is clearly equivalent to the condition of Proposition 3.8. Thus, for $q$-Gaussian processes the apriori possibly different definitions for "Markovianity" are all equivalent.

As Proposition 3.8. shows, Markovianity is a property of the underlying Hilbert space and does not depend on $q$. Thus we get as in the classical case the following characterization in terms of the covariance.

**Proposition 3.9.** *A $q$-Gaussian process with covariance $c$ is Markovian if and only if we have for all triples $s, u, t \in T$ with $s \leq u \leq t$ that*

$$c(t, s)c(u, u) = c(t, u)c(u, s).$$

*Proof.* See the proof of Theorem 3.9 in [Sim2]. □

**Corollary 3.10.** *The $q$-Brownian motion $(X_t^{qBM})_{t\in[0,\infty)}$, the $q$-Brownian bridge $(X_t^{qBB})_{t\in[0,1]}$, and the $q$-Ornstein-Uhlenbeck process $(X_t^{qOU})_{t\in\mathbb{R}}$ are all Markovian.*

Analogously, we have all statements of the classical Gaussian processes which depend only on Hilbert space properties. Let us just state the characterization of the Ornstein-Uhlenbeck process as the only stationary Gaussian Markov process with continuous covariance and the characterization of martingales among the Gaussian processes.

**Proposition 3.11.** *Let $(X_t)_{t\in T}$ be a $q$-Gaussian process which is stationary, Markovian and whose covariance $c(s, t) = c'(t - s)$ is continuous. Then $X_t = \alpha X_{\beta t}^{qOU}$ for suitable $\alpha, \beta > 0$.*

*Proof.* See the proof of the analogous statement for classical Gaussian processes, Corollary 4.10 in [Sim2]. □

**Definition 3.12.** *Let $(X_t)_{t\in T}$ be a stochastic process on a probability space $(\mathcal{A}, \varphi)$ and let the notations be as in Definition 3.7. Then we say that $(X_t)_{t\in T}$ is a martingale if*

$$\varphi[X_t|\mathcal{A}_{s]}] = X_s \qquad \text{for all } s \leq t.$$

**Proposition 3.13.** *A $q$-Gaussian process is a martingale if and only if $P_{s]}f_t = f_s$ for all $s \leq t$ – which is the case if and only if $c(s, t) = c(s, s)$ for all $s \leq t$.*

*Proof.* We have

$$\omega(f_s) = X_s = \mathrm{E}[X_t|\mathcal{A}_{s]}] = \Gamma_q(P_{s]})\omega(f_t) = \omega(P_{s]}f_t),$$

implying $P_{s]}f_t = f_s$. □

## 4. Classical Aspects of $q$-Gaussian Processes

In this section we want to address the question whether our non-commutative stochastic processes can also be interpreted classically.

**Definition 4.1.** *Let $(X_t)_{t\in T}$ be a stochastic process on some non-commutative probability space $(\mathcal{A}, \varphi)$. We call a classical real-valued process $(\tilde{X}_t)_{t\in T}$ on some classical probability space $(\Omega, \mathfrak{A}, P)$ a* **classical version** *of $(X_t)_{t\in T}$ if all time-ordered moments of $(X_t)_{t\in T}$ and $(\tilde{X}_t)_{t\in T}$ coincide, i.e. if we have for all $n \in \mathbb{N}$, all $t_1 \ldots, t_n \in T$ with $t_1 \leq \ldots \leq t_n$, and all bounded Borel functions $h_1, \ldots, h_n$ on $\mathbb{R}$ the equality*

$$\varphi[h_1(X_{t_1}) \ldots h_n(X_{t_n})] = \int_\Omega h_1(\tilde{X}_{t_1}(\omega)) \ldots h_n(\tilde{X}_{t_n}(\omega))dP(\omega).$$

*Remark 4.2.* Most calculations in a non-commutative context involve only time-ordered moments, see, e.g., the calculation of the Green function of the $q$-Ornstein-Uhlenbeck process in [NSp]. Thus, results of such calculations can also be interpreted as results for the classical version – if such a version exists.

It is clear that there is at most one classical version for a given non-commutative process $(X_t)_{t\in T}$. The problem consists in showing the existence. If we denote by $\mathbf{1}_B$ the characteristic function of a measurable subset $B$ of $\mathbb{R}$, then we can construct the classical version $(\tilde{X}_t)_{t\in T}$ of $(X_t)_{t\in T}$ via Kolmogorov's existence theorem from the collection of all $\mu_{t_1,\dots,t_n}$ ($n \in \mathbb{N}$, $t_1 \leq \dots \leq t_n$) – which are for $B_1, \dots, B_n \subset \mathbb{R}$ defined by

$$\begin{aligned}\mu_{t_1,\dots,t_n}(B_1 \times \dots \times B_n) &= P(\tilde{X}_{t_1} \in B_1, \dots, \tilde{X}_{t_n} \in B_n) \\ &= \varphi[\mathbf{1}_{B_1}(X_{t_1}) \dots \mathbf{1}_{B_n}(X_{t_n})]\end{aligned}$$

– if and only if all $\mu_{t_1,\dots,t_n}$ are probability measures. Whereas this is of course the case for $\mu_{t_1}$ and, in our tracial frame because of

$$\mu_{t_1,t_2}(B_1 \times B_2) = \varphi[\mathbf{1}_{B_1}(X_{t_1})\mathbf{1}_{B_2}(X_{t_2})] = \varphi[\mathbf{1}_{B_1}(X_{t_1})\mathbf{1}_{B_2}(X_{t_2})\mathbf{1}_{B_1}(X_{t_1})],$$

also for $\mu_{t_1,t_2}$, there is no apriori reason why it should be true for bigger $n$. And in general it is not. It is essentially the content of Bell's inequality that there are examples of non-commutative processes which possess no classical version – for a discussion of these subjects see, e.g., [KM].

But for special classes of non-commutative processes classical versions might exist. One prominent example of such a class are the Markov processes.

**Definition 4.3.** *Let $(X_t)_{t\in T}$ be a Markov process on a probability space $(\mathcal{A}, \varphi)$. Let, for $t \in T$,* spect$(X_t)$ *and $\nu_t$ be the spectrum and the distribution, respectively, of the self-adjoint operator $X_t$. Denote by*

$$L^\infty(X_t) := \mathrm{vN}(X_t) = L^\infty(\mathrm{spect}(X_t), \nu_t).$$

*The operators*

$$\mathcal{K}_{s,t} : L^\infty(X_t) \to L^\infty(X_s) \qquad (s \leq t),$$

*determined by*

$$\varphi[h(X_t)|\mathcal{A}_{s]}] = \varphi[h(X_t)|\mathcal{A}_{[s]}] = (\mathcal{K}_{s,t}h)(X_s)$$

*are called* **transition operators** *of the process $(X_t)_{t\in T}$, and, looked upon from the other side, the process $(X_t)_{t\in T}$ is called a* **dilation** *of the transition operators $\mathcal{K} = (\mathcal{K}_{s,t})_{s\leq t}$.*

The following theorem is by now some kind of folklore in quantum probability, see, e.g. [AFL, Kum2, BP, Bia1]. We just indicate the proof for sake of completeness.

**Theorem 4.4.** *If $(X_t)_{t\in T}$ is a Markov process on some probability space $(\mathcal{A}, \varphi)$, then there exists a classical version $(\tilde{X}_t)_{t\in T}$ of $(X_t)_{t\in T}$, which is a classical Markov process.*

*Proof.* One can express the time-ordered moments of a Markov process in terms of the transition operators via

$$\begin{aligned}\varphi[h_1(X_{t_1}) \dots h_n(X_{t_n})] &= \varphi[h_1(X_{t_1}) \dots h_n(X_{t_n})|\mathcal{A}_{t_{n-1}]}] \\ &= \varphi[h_1(X_{t_1}) \dots h_{n-1}(X_{t_{n-1}})\varphi[h_n(X_{t_n})|\mathcal{A}_{t_{n-1}]}]] \\ &= \varphi[h_1(X_{t_1}) \dots h_{n-1}(X_{t_{n-1}})(\mathcal{K}_{t_{n-1},t_n}h_n)(X_{t_{n-1}})] \\ &= \varphi[h_1(X_{t_1}) \dots h_{n-2}(X_{t_{n-2}})(h_{n-1} \cdot \mathcal{K}_{t_{n-1},t_n}h_n)(X_{t_{n-1}})] \\ &= \dots \\ &= \varphi[(h_1 \cdot \mathcal{K}_{t_1,t_2}(h_2 \cdot \mathcal{K}_{t_2,t_3}(h_3 \cdot \dots)))(X_{t_1})],\end{aligned}$$

from which it follows – because $\mathcal{K}_{s,t}$ preserves positivity – that the corresponding $\mu_{t_1,\ldots,t_n}$ are probability measures. That the classical version is also a classical Markov process follows by the same formula. □

**Corollary 4.5.** *There exist classical versions of all $q$-Gaussian Markov processes. In particular, we have classical versions of the $q$-Brownian motion, of the $q$-Brownian bridge, and of the $q$-Ornstein-Uhlenbeck process.*

Our aim now is to describe these classical versions more explicitly by calculating their transition probabilities in terms of the orthogonalizing measure $\nu_q$ and the kernel $p_r^{(q)}(x,y)$ of Theorem 1.10.

**Theorem 4.6.** *Let $(X_t)_{t\in T}$ be a $q$-Gaussian Markov process with covariance $c$ and put*

$$\lambda_t := \sqrt{c(t,t)} \qquad \text{and} \qquad \lambda_{s,t} := \frac{c(t,s)}{\sqrt{c(s,s)c(t,t)}}.$$

*1) We have*

$$L^\infty(X_t) = L^\infty([-2\lambda_t/\sqrt{1-q}, 2\lambda_t/\sqrt{1-q}], \nu_q(dx/\lambda_t)).$$

*2) If $\lambda_{s,t} = \pm 1$, then the transition operator $\mathcal{K}_{s,t}^{(q)}$ is given by*

$$(\mathcal{K}_{s,t}^{(q)} h)(x) = h(\pm x\lambda_t/\lambda_s).$$

*If $|\lambda_{s,t}| < 1$, then the transition operator $\mathcal{K}_{s,t}^{(q)}$ is given by*

$$(\mathcal{K}_{s,t}^{(q)} h)(x) = \int h(y) k_{s,t}^{(q)}(x,dy),$$

*where the transition probabilities $k_{s,t}^{(q)}$ are Feller kernels which have the explicit form*

$$k_{s,t}^{(q)}(x,dy) = p_{\lambda_{s,t}}^{(q)}(x/\lambda_s, y/\lambda_t)\nu_q(dy/\lambda_t).$$

*In particular, for $q = 0$ and $|\lambda_{s,t}| < 1$, we have the following transition probabilities for the free Gaussian Markov processes:*

$$k_{s,t}^{(0)}(x,dy) = \frac{1}{2\pi\lambda_t^2} \frac{(1-\lambda_{s,t}^2)\sqrt{4\lambda_t^2 - y^2}\,dy}{(1-\lambda_{s,t}^2)^2 - \lambda_{s,t}(1+\lambda_{s,t}^2)(x/\lambda_s)(y/\lambda_t) + \lambda_{s,t}^2((x^2/\lambda_s^2) + (y^2/\lambda_t^2))}.$$

Recall that a kernel $k(x,dy)$ is called Feller, if the map $x \mapsto k(x,dy)$ is weakly continuous and $k(x,\cdot) \to 0$ weakly as $x \to \pm\infty$ – or equivalently that the corresponding operator $\mathcal{K}$ sends $C_0(\mathbb{R})$ to $C_0(\mathbb{R})$, see, e.g., [DM].

*Proof.* 1) This was shown in [BSp2]; noticing the connection between $q$-relations and $q$-Hermite polynomials the assertion reduces essentially to part 1) of Theorem 1.10.
2) By Prop. 2.9, we know

$$\Psi(f^{\otimes n}) = \|f\|^n H_n^{(q)}(\omega(f)/\|f\|).$$

Let our $q$-Gaussian process $(X_t)_{t\in T}$ now be of the form $X_t = \omega(f_t)$. Markovianity implies

$$P_{s]}f_t = \mu f_s, \qquad \text{where} \qquad \mu = \frac{\langle f_t, f_s\rangle}{\langle f_s, f_s\rangle} = \frac{c(t,s)}{c(s,s)}.$$

Because of

$$\mathrm{E}[\,\Psi(f_t^{\otimes n})|\mathcal{A}_{s]}] = \Psi((P_{s]}f_t)^{\otimes n}) = \mu^n\,\Psi(f_s^{\otimes n})$$

we obtain with

$$\lambda_t := \|f_t\| = \sqrt{c(t,t)} \qquad \text{and} \qquad \lambda_{s,t} := \mu\frac{\lambda_s}{\lambda_t} = \frac{c(t,s)}{\sqrt{c(s,s)c(t,t)}}$$

the formula

$$\begin{aligned}\mathrm{E}[H_n^{(q)}(X_t/\lambda_t)|\mathcal{A}_{s]}] &= \tfrac{1}{\lambda_t^n}\mathrm{E}[\,\Psi(f_t^{\otimes n})|\mathcal{A}_{s]}]\\ &= \tfrac{\mu^n}{\lambda_t^n}\,\Psi(f_s^{\otimes n})\\ &= (\mu\tfrac{\lambda_s}{\lambda_t})^n H_n^{(q)}(X_s/\lambda_s)\\ &= \lambda_{s,t}^n H_n^{(q)}(X_s/\lambda_s),\end{aligned}$$

implying

$$\mathcal{K}_{s,t}^{(q)}(H_n^{(q)}(\cdot/\lambda_t)) = \lambda_{s,t}^n H_n^{(q)}(\cdot/\lambda_s).$$

Let us now consider the canonical extension of our transition operators from the $L^\infty$-spaces to the $L^2$-spaces, i.e.

$$\mathcal{K}_{s,t}^{(q)} : L^2(X_t) \to L^2(X_s).$$

If we use the fact that the rescaled $q$-Hermite polynomials $(H_n^{(q)}(\cdot/\lambda_t)/\sqrt{[n]!})_{n\in\mathbb{N}_0}$ constitute an orthonormal basis of $L^2(X_t)$, we get directly the assertion in the case $\lambda_{s,t} = \pm 1$. (For $\lambda_{s,t} = -1$ one also has to note that $H_{2k}^{(q)}$ and $H_{2k+1}^{(q)}$ are even and odd polynomials, respectively.)

In the case $|\lambda_{s,t}| < 1$, our formula implies that $\mathcal{K}_{s,t}^{(q)}$ is a Hilbert-Schmidt operator, thus it has a concrete representation by a kernel $k_{s,t}^{(q)}$, which is given by

$$\begin{aligned}k_{s,t}^{(q)}(x,dy) &= \textstyle\sum_{n=0}^\infty \frac{\lambda_{s,t}^n}{[n]_q!} H_n^{(q)}(x/\lambda_s)H_n^{(q)}(y/\lambda_t)\nu_q(dy/\lambda_t)\\ &= p_{\lambda_{s,t}}(x/\lambda_s, y/\lambda_t)\nu_q(dy/\lambda_t).\end{aligned}$$

That our kernels are Feller follows from the fact that, by Theorem 2.13, our second quantization (i.e. our transition operators) restrict to the $C^*$-level (i.e. to continuous functions).

The formula for $k_{s,t}^{(0)}$ follows from the concrete form of $p_r^{(0)}$ of Theorem 1.10 and the fact that

$$\nu_0(dy) = \frac{1}{2\pi}\sqrt{4-y^2}dy \qquad \text{for} \qquad y \in [-2,2]. \qquad \square$$

The main formula of our proof, namely the action of the conditional expectation on the $q$-Hermite polynomials, says that we have some quite canonical martingales associated to $q$-Gaussian Markov processes – provided the factor $\lambda_{s,t}$ decomposes into a quotient $\lambda_{s,t} = \lambda(s)/\lambda(t)$. Since this can be assured by a corresponding factorization property of the covariance function – which is not very restrictive for Gaussian Markov processes, see Theorem 4.9 of [Sim2] – we get the following corollary.

**Corollary 4.7.** *Let $(X_t)_{t\in T}$ be a $q$-Gaussian process whose covariance factorizes for suitable functions $g$ and $f$ as*

$$c(s,t) = g(s)f(t) \qquad \text{for } s \le t.$$

*Then, for all $n \in \mathbb{N}_0$, the processes $(M_n(t))_{t\in T}$ with*

$$M_n(t) := \big(g(t)/f(t)\big)^{n/2} H_n^{(q)}(X_t/\lambda_t)$$

*are martingales.*

Note that the assumption on the factorization of the covariance is in particular fulfilled for the $q$-Brownian motion, for the $q$-Ornstein-Uhlenbeck process, and for the $q$-Brownian bridge.

*Proof.* Our assumption on the covariance implies

$$\lambda_{s,t} = \sqrt{\frac{g(s)/f(s)}{g(t)/f(t)}},$$

hence our formula for the action of the conditional expectation on the $q$-Hermite polynomials (in the proof of Theorem 4.6) can be written as

$$\big(g(t)/f(t)\big)^{n/2} \mathrm{E}[H_n^{(q)}(X_t/\lambda_t)|\mathcal{A}_{s]}] = \big(g(s)/f(s)\big)^{n/2} H_n^{(q)}(X_s/\lambda_s),$$

which is exactly our assertion. □

*Remark 4.8.* Consider the $q$-Brownian motion $(X_t^{qBM})_{t\in[0,\infty)}$. Then the corollary states that

$$M_n^{(q)}(t) := t^{n/2} H_n^{(q)}(X_t^{qBM}/\sqrt{t})$$

is a martingale. In terms of quantum stochastic integrals these martingales would have the form

$$M_n^{(q)}(t) = \underset{\substack{0\le t_1,\dots,t_n\le t\\ t_i\ne t_j\ (i\ne j)}}{\int\cdots\int} dX_{t_1}^{qBM}\dots dX_{t_n}^{qBM}.$$

Since at the moment, for general $q$, no rigorous theory of $q$-stochastic integration exists, this has to be taken as a purely formal statement. For $q = 0$, however, such a rigorous theory was developed in [KSp], and the above representation by stochastic integrals was established by Biane [Bia2]. In this case, he could put this representation into the form of the stochastic differential equation

$$M_n^{(0)}(t) = \sum_{k=0}^{n-1}\int_0^t M_k^{(0)}(s)dX_s^{0BM} M_{n-k-1}^{(0)}(s),$$

which should be compared with the classical formula

$$M_n^{(1)}(t) = n\int_0^t M_{n-1}^{(1)}(s)dX_s^{1BM}.$$

*Example 4.9 (Free Gaussian processes).* We will now specialize the formula for $k^{(0)}_{s,t}$ to the case of the free Brownian motion, the free Ornstein-Uhlenbeck process and the free Brownian bridge. The transition probabilities for the two former cases were also derived by Biane [Bia1] in the context of processes with free increments.
1) Free Brownian motion: We have $c(s,t) = \min(s,t)$, thus

$$\lambda_t = \sqrt{t} \qquad \text{and} \qquad \lambda_{s,t} = \sqrt{s/t}.$$

This yields

$$k_{s,t}(x,dy) = \frac{(t-s)}{(t-s)^2 - (t+s)xy + x^2 t + y^2 s} \frac{\sqrt{4t-y^2}dy}{2\pi}$$

for

$$x \in [-2\sqrt{s}, 2\sqrt{s}] \qquad \text{and} \qquad y \in [-2\sqrt{t}, 2\sqrt{t}].$$

2) Free Ornstein-Uhlenbeck process: We have $c(s,t) = e^{-|t-s|}$, thus

$$\lambda_t = 1 \qquad \text{and} \qquad \lambda_{s,t} = e^{-|t-s|}.$$

Since this process is stationary, it suffices to consider the transition probabilities for $s = 0$:

$$k_{0,t}(x,dy) = \frac{(e^{2t}-1)}{4\sinh^2 t - 2xy\cosh t + x^2 + y^2} \frac{\sqrt{4-y^2}dy}{2\pi} \qquad \text{for} \qquad x,y \in [-2,2].$$

Let us also calculate the generator $N$ of this process – which is characterized by

$$\mathcal{K}_{s,t} = e^{-(t-s)N}.$$

It has the property

$$NH^{(0)}_n = nH^{(0)}_n \qquad (n \in \mathbb{N}_0),$$

and differentiating the above kernel shows that it should be given formally by a kernel $-2/(y-x)^2$ with respect to $\nu_0$. Making this more rigorous [vWa] yields that $N$ has on functions which are differentiable the form

$$(Nh)(x) = xf'(x) - 2\int \frac{f(y) - f(x) - f'(x)(y-x)}{(y-x)^2}\nu_0(dy).$$

3) Free Brownian bridge: We have $c(s,t) = s(1-t)$ for $s \le t$, thus

$$\lambda_t = \sqrt{t(1-t)} \qquad \text{and} \qquad \lambda_{s,t} = \sqrt{\frac{s(1-t)}{t(1-s)}}.$$

This yields

$$k_{s,t}(x,dy) =$$
$$= \frac{1-s}{1-t} \frac{(t-s)}{(t-s)^2 - (s+t-2st)xy + t(1-t)x^2 + s(1-s)y^2} \frac{\sqrt{4t(1-t)-y^2}dy}{2\pi},$$

for

$$x \in [-2\sqrt{s(1-s)}, 2\sqrt{s(1-s)}] \qquad \text{and} \qquad y \in [-2\sqrt{t(1-t)}, 2\sqrt{t(1-t)}].$$

*Example 4.10 (Fermionic Gaussian processes).* For illustration, we also want to consider the fermionic ($q = -1$) analogue of Gaussian processes. Although this case has not been included in our frame everything works similarly, the only difference is that in the Fock space we get a kernel of our scalar product consisting of anti-symmetric tensors. This is responsible for the fact that the corresponding $(-1)$-Hermite polynomials collapse just to

$$H_0^{(-1)}(x) = 1 \qquad \text{and} \qquad H_1^{(-1)}(x) = x.$$

The corresponding measure $\nu_{-1}$ is not absolutely continuous with respect to the Lebesgue measure anymore, but collapses to

$$\nu_{-1}(dx) = \frac{1}{2}(\delta_{-1}(dx) + \delta_{+1}(dx)).$$

This yields

$$p_r^{(-1)}(x,y) = H_0^{(-1)}(x)H_0^{(-1)}(y) + rH_1^{(-1)}(x)H_1^{(-1)}(y) = 1 + rxy,$$

giving as transition probabilities

$$k_{s,t}^{(-1)}(x,dy) = \frac{1}{2}(1 + \frac{c(s,t)}{c(s,s)c(t,t)}xy)(\delta_{-\sqrt{c(t,t)}}(dy) + \delta_{+\sqrt{c(t,t)}}(dy)).$$

1) Fermionic Brownian motion: $X_t$ can only assume the values $+\sqrt{t}$ and $-\sqrt{t}$ and the transition probabilities are given by the table

$$\begin{array}{ccc} k_{s,t} & \sqrt{t} & -\sqrt{t} \\ \sqrt{s} & \frac{1}{2}(1+\sqrt{s/t}) & \frac{1}{2}(1-\sqrt{s/t}) \\ -\sqrt{s} & \frac{1}{2}(1-\sqrt{s/t}) & \frac{1}{2}(1+\sqrt{s/t}) \end{array}.$$

This case coincides with the corresponding $c = -1$ case of the Azéma martingale, see [Par1].

2) Fermionic Ornstein-Uhlenbeck process: This stationary process lives on the two values $+1$ and $-1$ with the following transition probabilities

$$\begin{array}{ccc} k_{s,t} & 1 & -1 \\ 1 & \frac{1}{2}(1+e^{-(t-s)}) & \frac{1}{2}(1-e^{-(t-s)}) \\ -1 & \frac{1}{2}(1-e^{-(t-s)}) & \frac{1}{2}(1+e^{-(t-s)}) \end{array}.$$

This classical two state Markov realization of the corresponding fermionic relations has been known for a long time, see [FB].

3) Fermionic Brownian bridge: $X_t$ can only assume the values $+\sqrt{t(1-t)}$ and $-\sqrt{t(1-t)}$ and the transition probabilities are given by the table

$$\begin{array}{ccc} k_{s,t} & \sqrt{t(1-t)} & -\sqrt{t(1-t)} \\ \sqrt{s(1-s)} & \frac{1}{2}(1+\sqrt{\frac{s(1-t)}{t(1-s)}}) & \frac{1}{2}(1-\sqrt{\frac{s(1-t)}{t(1-s)}}) \\ -\sqrt{s(1-s)} & \frac{1}{2}(1-\sqrt{\frac{s(1-t)}{t(1-s)}}) & \frac{1}{2}(1+\sqrt{\frac{s(1-t)}{t(1-s)}}) \end{array}.$$

*Example 4.11 (Hypercontractivity).* Consider the $q$-Ornstein-Uhlenbeck process with stationary transition operators $\mathcal{K}_t^{(q)} := \mathcal{K}_{s,s+t}^{qOU}$. Note that this $q$-Ornstein-Uhlenbeck semigroup is nothing but the second quantization of the simplest contraction, namely with the one-dimensional real Hilbert space $\mathcal{H} = \mathbb{R}$ and the corresponding identity operator $\mathbf{1} : \mathbb{R} \to \mathbb{R}$ we have

$$\Gamma_q(\mathbb{R}) \cong L^\infty([-2/\sqrt{1-q}, 2/\sqrt{1-q}], \nu_q(dx)) \qquad \text{and} \qquad \Gamma_q(e^{-t}\mathbf{1}) \cong \mathcal{K}_t^{(q)}.$$

We have seen that the $\mathcal{K}_t^{(q)}$ are, for all $t > 0$, contractions on $L^2$ and on $L^\infty$ (and thus, by duality and interpolation, on all $L^p$). In the classical case $q = 1$ (and also for $q = -1$) it is known [Sim1, Nel1, Nel2, Gro, CL] that much more is true, namely the Ornstein-Uhlenbeck semigroup is also hypercontractive, i.e. it is bounded as a map from $L^2$ to $L^4$ for sufficiently large $t$. Having the concrete form of the kernel

$$k_t^{(q)}(x, dy) = p_{e^{-t}}^{(q)}(x, y)\nu_q(dy)$$

of $\mathcal{K}_t^{(q)}$, it is easy to check that we also have hypercontractivity for all $-1 < q < 1$. Even more, we can show that $\mathcal{K}_t^{(q)}$ is bounded from $L^2$ to $L^\infty$ for $t > 0$, i.e. we have what is called "ultracontractivity" [Dav] – which is, of course, not given for $q = \pm 1$. This ultracontractivity follows from the estimate

$$\|\mathcal{K}_t^{(q)} h\|_\infty \leq \alpha(t,q)^{1/2}\|h\|_2 \qquad \text{where} \qquad \alpha(t,q) := \sup_{x\in[-2,2]} \sup_{y\in[-2,2]} p_{e^{-t}}^{(q)}(x,y),$$

and from the explicit form of $p_r^{(q)}$ from Theorem 1.10, which ensures that $\alpha(t,q)$ is finite for $t > 0$ and $-1 < q < 1$ (comp. also [Dav], Lemma 2.1.2). One may also note that for small $t$ the leading term of $\alpha(t,q)^{1/2}$ is of order $t^{-3/2}$.

*4.12 Open Problems*. 1) The situation concerning classical versions of non-Markovian $q$-Gaussian processes is not clear at the moment.
2) Consider a symmetric measure $\mu$ on $\mathbb{R}$ with compact support. Then there exist a sequence of polynomials $(P_n)_{n\in\mathbb{N}_0}$ such that $P_n$ is of degree $n$ and such that these polynomials are orthogonal with respect to $\mu$. Let us define a semigroup $U_t$ on $L^2(\mu)$ by

$$U_t P_n = e^{-nt} P_n.$$

If these $U_t$ are positivity preserving then they constitute the transition operators of a stationary Markov process, whose stationary distribution is given by $\mu$. Our $q$-Ornstein-Uhlenbeck process is an example of this general construction for the measure $\nu_q$. The existence of the functor $\Gamma_q$ "explains" the fact that the $q$-Ornstein-Uhlenbeck semigroup is positivity preserving from a more general (non-commutative) point of view – note that although Theorem 2.11 is for dim $\mathcal{H}$ = dim $\mathcal{H}' = 1$ a purely commutative statement, its proof is even in this case definitely non-commutative. Of course, not for all measures $\mu$ the semigroup $U_t$ is positivity preserving. But one might wonder whether it is possible to find for each measure with this property –or at least for some special class of such measures – some analogous kind of functor. See also [BSp4] for related investigations.

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## References


[AFL] Accardi, L., Frigerio, A., Lewis, J.T.: Quantum stochastic processes. Publ. RIMS **18**, 97–133 (1982)

[BP] Bhat, B.V.R., Parthasarathy, K.R.: Markov dilations of nonconservative dynamical semigroups and a quantum boundary theory. Ann. Inst. Henri Poincaré **31**, 601–651 (1995)

[Bia1] Biane, Ph.: On processes with free increments. Mfath. Z. To appear

[Bia2] Biane, Ph.: Free Brownian motion, free stochastic calculus and random matrices. In: *Free probability theory* (Fields Institute Communications 12), ed. D. Voiculescu, Providence: AMS, 1997, pp. 1–19

[Bia3] Biane, Ph.: Quantum Markov processes and group representations. Preprint, 1995

[BSp1] Bożejko, M., Speicher, R.: An example of a generalized Brownian motion. Commun. Math. Phys. **137**, 519–531 (1991)

[BSp2] Bożejko, M., Speicher, R.: An example of a generalized Brownian motion II. *Quantum Probability and Related Topics VII*, ed. L. Accardi. Singapore: World Scientific, 1992, pp. 219–236

[BSp3] Bożejko, M., Speicher, R.: Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces. Math. Ann. **300**, 97–120 (1994)

[BSp4] Bożejko, M., Speicher, R.: Interpolations between bosonic and fermionic relations given by generalized Brownian motions. Math. Z. **222**, 135–160 (1996)

[Bre] Bressoud, D.M.: A simple proof of Mehler's formula for $q$-Hermite polynomials. Indiana Univ. Math. J. **29**, 577–580 (1980)

[CL] Carlen, E.A., Lieb, E.H.: Optimal hypercontractivity for Fermi fields and related non-commutative integration inequalities. Commun. Math. Phys. **155**, 27–46 (1993)

[Cun] Cuntz, J.: Simple $C^*$-algebras generated by isometries. Commun. Math. Phys **57**, 173–185 (1977)

[Dav] Davies, E.B.: *Heat kernels and spectral theory*, Cambridge: Cambridge University Press, 1989

[DM] Dellacherie, D., Meyer, P.A.: *Probabilités et potentiel*, Paris: Hermann, 1975

[DN] Dykema, K., Nica, A.: On the Fock representation of the $q$-commutation relations. J. reine angew. Math. **440**, 201–212 (1993)

[Eva] Evans, D.E.: On $O_n$. Publ. RIMS **16**, 915–927 (1980)

[Fiv] Fivel, D.I.: Interpolation between Fermi and Bose statistics using generalized commutators. Phys. Rev. Lett. **65**, 3361–3364 (1990), Erratum **69**, 2020 (1992)

[FB] Frisch, U., Bourret, R.: Parastochastics. J. Math. Phys. **11**, 364–390 (1970)

[GR] Gasper, G., Rahman, M.: *Basic hypergeometric functions*, Cambridge: Cambridge U.P., 1990

[Gre] Greenberg, O.W.: Particles with small violations of Fermi or Bose statistics. Phys. Rev. D **43**, 4111–4120 (1991)

[Gro] Gross, L.: Existence and uniqueness of physical ground states. J. Funct. Anal. **10**, 52–109 (1972)

[Hal] Halmos, P.R.: Normal dilations and extensions of operators. Summa Brasiliensis Math. **2**, 125–134 (1950)

[HP] Hudson, R.L., Parthasarathy, K.R.: Quantum Ito's formula and stochastic evolution. Commun. Math. Phys. **93**, 301–323 (1984)

[ISV] Ismail, E.M., Stanton, D., Viennot, G.: The combinatorics of $q$-Hermite polynomials and Askey-Wilson integral. Europ. J. Comb. **8**, 379–392 (1987)

[JSW1] Jørgensen, P.E.T., Schmitt, L.M., Werner, R.F.: $q$-canonical commutation relations and stability of the Cuntz algebra. Pac. J. Math. **165**, 131–151 (1994)

[JSW2] Jørgensen, P.E.T., Schmitt, L.M., Werner, R.F.: Positive representations of general commutation relations allowing Wick ordering. J. Funct. Anal. **134**, 3–99 (1995)

[JW] Jørgensen, P.E.T., Werner, R.F.: Coherent states on the $q$-canonical commutation relations. Commun. Math. Phys. **164**, 455–471 (1994)

[Kum1] Kümmerer, B.: Markov dilations on $W^*$-algebras. J. Funct. Anal. **63**, 139–177 (1985)

[Kum2] Kümmerer, B.: Survey on a theory of non-commutative stationary Markov processes. *Quantum Probability and Applications III*, L. Accardi, W.v. Waldenfels, ed., Berlin: Springer-Verlag, 1988, pp. 228–244

[KM] Kümmerer, B., Maassen, H.: Elements of Quantum Probability. *Quantum Probability Communications X.* To appear

[KSp] Kümmerer, B., Speicher, R.: Stochastic integration on the Cuntz algebra $O_\infty$. J. Funct. Anal. **103**, 372–408 (1992)

[LM1] van Leeuwen, H., Maassen, H.: A $q$-deformation of the Gauss distribution. J. Math. Phys. **36**, 4743–4756 (1995)

[LM2] van Leeuwen, H., Maassen, H.: An obstruction for $q$-deformation of the convolution product. J. Phys. A **29**, 1–8 (1996)

[Mey] Meyer, P.A.: *Quantum probability for probabilists*, Lecture Notes in Mathematics **1538**, Heidelberg: Springer-Verlag, 1993
[Mol] Møller, J.S.: Second quantization in a quon-algebra. J. Phys. A **26**, 4643–4652 (1993)
[Nel1] Nelson, E.: Construction of quantum fields from Markoff fields. J. Funct. Anal. **12**, 97–112 (1973)
[Nel2] Nelson, E.: The free Markoff field. J. Funct. Anal. **12**, 211–227 (1973)
[NSp] Neu, P., Speicher, R.: Spectra of Hamiltonians with generalized single-site dynamical disorder. Z. Phys. B **95**, 101–111 (1994)
[Par1] Parthasarathy, K.R.: Azéma martingales and quantum stochastic calculus. *Proc. R. C. Bose Symposium*, Wiley Eastern, 1990, pp. 551–569
[Par2] Parthasarathy, K.R.: *An introduction to quantum stochastic calculus. Monographs in Mathematics* Vol. **85**, Basel: Birkhäuser, 1992
[Rog] Rogers, L.J.: On a three-fold symmetry in the elements of Heine series. Proc. London Math. Soc. **24**, 171–179 (1893)
[Sch] Schürmann, M.: Quantum $q$-white noise and a $q$-central limit theorem. Commun. Math. Phys. **140**, 589–615 (1991)
[Seg] Segal, I.E.: Tensor algebras over Hilbert spaces. Trans. Amer. Math. Soc. **81**, 106–134 (1956)
[Sim1] Simon, B.: *The $P(\phi)_2$ Euclidean (Quantum) Field Theory*. Princeton, NJ: Princeton University Press, 1974
[Sim2] Simon, B.: *Functional Integration and Quantum Physics*. New York: Academic Press, 1979
[Spe1] Speicher, R.: Generalized statistics of macroscopic fields. Lett. Math. Phys. **27**, 97–104 (1993)
[Spe2] Speicher, R.: On universal products. In: "Free probability theory" (Fields Institute Communications 12), ed. D. Voiculescu. Providence: AMS, 1997, pp. 257–266
[Sta] Stanciu, S.: The energy operator for infinite statistics. Commun. Math. Phys. **147**, 211–216 (1992)
[Sze] Szego, G.: Ein Beitrag zur Theorie der Thetafunktionen. Sitz. Preuss. Akad. Wiss. Phys. Math. Ll **19**, 242–252 (1926)
[Tak] Takesaki, M.: Conditional expectations in von Neumann algebras. J. Funct. Anal. **9**, 306–321 (1972)
[Voi] Voiculescu, D.: Symmetries of some reduced free product $C^*$-algebras. In: Operator Algebras and their Connection with Topology and Ergodic Theory, Lecture Notes in Mathematics Vol. **1132** Heidelberg: Springer-Verlag, 1985, pp. 556–588
[VDN] Voiculescu, D., Dykema, K., Nica, A.: *Free Random Variables*. Providence, RI: AMS, 1992
[vWa] Waldenfels, W. von: Fast positive Operatoren. Z. Wahrscheinlichkeitstheorie verw. Geb. **4**, 159–174 (1965)
[Wer] Werner, R.F.: The free quon gas suffers Gibb's paradox. Phys. Rev. D **48**, 2929–2934 (1993)
[Wil] Wilde, I.F.: The free Fermi field as a Markov field. J. Funct. Anal. **15**, 12–21 (1974)
[YW] Yu, T., Wu, Z.-Y.: Construction of the Fock-like space for quons. Science in China (Series A) **37**, 1472–1483 (1994)
[Zag] Zagier, D.: Realizability of a model in infinite statistics. Commun. Math. Phys. **147**, 199–210 (1992)

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