The Conditional Central Limit Theorem For a Superlinear Process

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Outline

- Superlinear Processes
- The Theorem and Examples
- Some Background
- About the Proof

Super-linear Processes

Notation • J is a countable set

- $F_j, \ j \in J$ are DFs with mean 0 and unit variance
- $\xi_{i,j} \sim F_j, \ i \in \mathbb{Z}, \ j \in J$, are independent.
- $c_{i,j}$, are square summable.

Then

$$X_k = \sum_{j \in J} \sum_{i \in \mathbb{Z}} c_{i,j} \xi_{k-i,j}$$

is a **super-linear process**. Let

$$S_n = X_1 + \cdots + X_n.$$

Herndorff's Theorem

(1984, *Ann. Stat.*)

There is a strongly mixing super-linear process for which,

$$E(X_i X_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

and

$$\liminf_{n\to\infty} P\left[S_n = 0\right] \geq \frac{1}{2}.$$

In the Proof:

- $\xi_{i,j}$ have large values for large j
- $c_{i,j}$ have finite range and $\sum_{i\in\mathbb{Z}} c_{i,j} = 0$ for each j

Causal Processes

If $c_{i,j} = 0$ for $i < 0$, then

$$X_k = \sum_{j\in J}\sum_{i=0}^{\infty} c_{i,j}\xi_{k-i,j},$$

and the process is said to **causal**. Let

$$b_{n,j} = c_{0,j} + \cdots + c_{n,j},$$
$$\bar{b}_{n,j} = (b_{n,1} + \cdots + b_{n,n})/n,$$
$$\mathbf{b}_n = (b_{n,j} : J \in J) \in \ell^2(J) \quad \text{and} \quad \bar{\mathbf{b}}_n = (\bar{b}_{n,j} : J \in J).$$

Then

$$S_n = \sum_{j\in J}\sum_{i\le 0}[b_{n-i,j} - b_{-i,j}]\xi_{i,j} + \sum_{j\in J}\sum_{i=1}^{n} b_{n-i,j}\xi_{i,j}.$$

Let

$$\mathcal{F}_k = \sigma\{\xi_{i,j} : i \le k,\ j \in J\}.$$

Conditional Normality

Suppose that

$$\sigma_n^2 := E(S_n^2) \to \infty$$

and let

$$\Phi_n(\omega; z) = P\left[\frac{S_n}{\sigma_n} \le z | \mathcal{F}_0\right](\omega).$$

Then the **conditional central limit theorem** holds if $\Phi_n \Rightarrow^p \Phi$, the standard normal DF.

Note: $\cdots X_{-1}, X_0, X_1, \cdots$ can be any adapted stationary sequence with mean 0 and finite variance.

Linear Processes

From WW (2004, AP)

Known Result: For a causal linear process ($J = \{0\}$), the CCLThm holds iff

$$\sum_{i=0}^{\infty}[b_{i+n} - b_i]^2 = o[\sum_{i=1}^{n} b_i^2],$$

where $b_i = b_{i,0}$, etc. $\cdots$.

Note: Condition on the coefficients

The Theorem

Theorem. For a causal super-linear process, the conditional central theorem holds iff

$$\sum_{j\in J}\sum_{i=0}^{\infty}[b_{n+i,j}-b_{i,j}]^2 = o(n\|\bar{\mathbf{b}}_n\|^2), \tag{s}$$

and

$$L_n^*(\epsilon) = \frac{1}{\|\bar{\mathbf{b}}_n\|^2}\sum_{j\in J}\bar{b}_{n,j}^2\int_{|\bar{b}_{n,j}z|>\epsilon\ \|\bar{\mathbf{b}}_n\|\sqrt{n}} z^2F_j\{dz\}\to 0 \tag{lf}$$

for each $\epsilon > 0$.

Notes 1. Recall $b_{n,j} = c_{0,j} + \cdots + c_{n,j}$, etc. $\cdots$.

2. Interplay in (*lf*).

Example

If F_j assigns mass 2^{-j-1} to $\pm 2^j$ and $1-2^{-j}$ to 0 for $j \geq 1$, then the CCLThm holds iff

$$\lim_{n\to\infty} \sum_{2^j|\bar{b}_{n,j}|>\epsilon\|\bar{\mathbf{b}}_n\|\sqrt{n}} \frac{\bar{b}_{n,j}^2}{\|\bar{\mathbf{b}}_n\|^2} = 0$$

for all $\epsilon > 0$.

Necessary Conditions

From WW (2004, AP)

Notation: • Let $\cdots X_{-1}, X_0, X_1, \cdots$ be any adapted stationary sequence with mean 0 and finite variance; and let $\|\cdot\|$ be the norm in $L^2(P)$.

- Recall that $\sigma_n^2 = \|S_n\|^2 \to \infty$.

Proposition. If the conditional central limit theorem holds, then

$$\|E(S_n|\mathcal{F}_0)\|^2 = o(\sigma_n^2); \qquad (*)$$

and if (*) holds, then

$$\sigma_n^2 = n\ell(n),$$

where ℓ is slowly varying.

Martingale Approximations

From WW (2004, AP)

For each $n \geq 1$, let $M_{n,k},\ k = 0, 1, 2, \cdots$ be a martingale adapted to $\mathcal{F}_k$. Then M_{nk} ia a **martingale approximation** iff

$$\max_{k \leq n} \|S_k - M_{n,k}\| = o(\sigma_n^2).$$

Write

$$D_{n,k} = M_{n,k} - M_{n,k-1}.$$

Martingale Approximations

Continued

Theorem

$$\|E(S_n|\mathcal{F}_0)\|^2 = o(\sigma_n^2) \qquad (*)$$

iff there is a martingale approximation M_{nk} with stationary increments in which case $\sigma_n^2 \sim nE(D_{n,1}^2)$ and $\ell(n) \sim E(D_{n,1}^2)$.

From the Proof. In the proof,

$$D_{n,1} = \frac{1}{n}\sum_{k=1}^{n}\left[E(S_k|\mathcal{F}_1) - E(S_k|\mathcal{F}_0)\right].$$

Necessary and Sufficient Conditions

From WW (2004, AP)

Theorem The CCLThm holds iff $\exists$ a martingale approximation for which

$$\frac{1}{\sigma_n^2}\sum_{k=1}^{n} E(D_{nk}^2|\mathcal{F}_{k-1}) \to^{\mathrm{p}} 1 \qquad (S)$$

$$L_n(\epsilon) := \frac{1}{\sigma_n^2}\sum_{k=1}^{n} E(D_{nk}^2\mathbf{1}_{\{|D_{nk}|\geq\epsilon\sigma_n\}}|\mathcal{F}_{k-1}) \to^{\mathrm{p}} 0 \qquad (LF)$$

forall $\epsilon > 0$.

The Theorem

Restated

Theorem. For a causal super-linear process, the conditional central theorem holds iff

$$\sum_{j\in J}\sum_{i=0}^{\infty}[b_{n+i,j}-b_{i,j}]^2 = o(n\|\bar{\mathbf{b}}_n\|^2), \qquad (s)$$

and

$$L_n^*(\epsilon) = \frac{1}{\|\bar{\mathbf{b}}_n\|^2}\sum_{j\in J}\bar{b}_{n,j}^2\int_{|\bar{b}_{n,j}z|>\epsilon\ \|\bar{\mathbf{b}}_n\|\sqrt{n}} z^2 F_j\{dz\} \to 0 \qquad (lf)$$

for each $\epsilon > 0$.

From the Proof

Some Algebra

$$S_n = \sum_{j\in J}\sum_{i\le 0}[b_{n-i,j} - b_{-i,j}]\xi_{i,j} + \sum_{j\in J}\sum_{i=1}^{n} b_{n-i,j}\xi_{i,j},$$

$$\sigma_n^2 = \sum_{j\in J}\left[\sum_{i=0}^{\infty}[b_{n+i,j} - b_{i,j}]^2 + \sum_{j\in J}\sum_{i=1}^{n} b_{n,n-i}^2\right],$$

$$E(S_n|\mathcal{F}_0) = \sum_{j\in J}\sum_{i\le 0}[b_{n-i,j} - b_{-i,j}]\xi_{i,j},$$

$$\|E(S_n|\mathcal{F}_0)\|^2 = \sum_{j\in J}\sum_{i=0}^{\infty}[b_{n+i,j} - b_{i,j}]^2,$$

So, (*) iff (s)

More From the Proof

More Algebra: From

$$E(S_n|\mathcal{F}_0) = \sum_{j\in J}\sum_{i\le 0}[b_{n-i,j} - b_{-i,j}]\xi_{i,j}$$

and

$$E(S_n|\mathcal{F}_1) = \sum_{j\in J}\sum_{i\le 1}[b_{n-i,j} - b_{-i,j}]\xi_{i,j},$$

$$D_{n,1} = \sum_{j\in J}\bar{b}_{n,j}\xi_{1,j} \quad \text{and} \quad D_{n,k} = \sum_{j\in J}\bar{b}_{n,j}\xi_{k,j}.$$

The Stability Conditions

From $D_{n,k} = \sum_{j\in J} \bar{b}_{n,j}\xi_{k,j}$,

$$\sum_{k=1}^{n} E(D_{nk}^2|\mathcal{F}_{k-1}) = \sum_{k=1}^{n} E(D_{nk}^2) = n\|\bar{\mathbf{b}}_n\|^2.$$

So,

$$\sum_{j\in J}\sum_{i=0}^{\infty}[b_{n+i,j} - b_{i,j}]^2 = o(\|\bar{\mathbf{b}}_n\|^2), \qquad (s)$$

iff

$$\sigma_n^2 \sim n\|\bar{\mathbf{b}}_n\|^2$$

iff

$$\frac{1}{\sigma_n^2}\sum_{k=1}^{n} E(D_{nk}^2) \to^{\mathrm{p}} 1 \qquad (S)$$

The Lindeberg Feller Conditions

$$L_n^*(\epsilon) = \frac{1}{\|\bar{\mathbf{b}}_n\|^2} \sum_{j\in J} \bar{b}_{n,j}^2 \int_{|\bar{b}_{n,j} z| > \epsilon \ \|\bar{\mathbf{b}}_n\| \sqrt{n}} z^2 F_j\{dz\} \to 0 \qquad (lf)$$

and

$$L_n(\epsilon) := \frac{1}{\sigma_n^2} \sum_{k=1}^{n} E(D_{nk}^2 \mathbf{1}_{\{|D_{nk}| \geq \epsilon\sigma_n\}} | \mathcal{F}_{k-1}) \to^{\mathrm{p}} 0 \qquad (LF)$$

Notes 1. As above

$$L_n(\epsilon) := \frac{1}{\sigma_n^2} \sum_{k=1}^{n} E(D_{nk}^2 \mathbf{1}_{\{|D_{nk}| \geq \epsilon\sigma_n\}})$$

2. Need to related

$$E[D_{nk}^2 \mathbf{1}_{\{|D_{nk}| \geq c\}}] \quad \text{to} \quad \sum_{j\in J} \int_{|x|>c} x^2 F_j\{dx\}.$$

Some Inequalities

The Baum Katz Inequalities:

$$P[|D_{n,k}| > 3x] \le \left(\frac{\|\bar{\mathbf{b}}_n\|^2}{x^2}\right)^2 + \sum_{j \in J} P|\bar{b}_{n,j}\xi_{k,j}| > x].$$

Integration by Parts.

$$L_n(\epsilon) \le 162\left(\frac{n\|\bar{\mathbf{b}}_n\|^4}{\epsilon^2\sigma_n^4}\right) + \frac{18n\|\bar{\mathbf{b}}_n\|^2}{\sigma_n^2} L_n^*(\frac{1}{4}\epsilon).$$

More Inequlities

Let $Z_j,\ j \in J$ be independent with mean 0 and square summable variances b_j^2,

$$Y = \sum_{j \in J} Z_j \quad \text{and} \quad Y_{-j} = \sum_{k \neq j} Z_k.$$

Then

$$\begin{aligned} P[Y \geq \frac{1}{2} Z_j | Z_j] &= P[Y_{-j} \geq -\frac{1}{2} Z_j | Z_j] \\ &\geq \frac{Z_j^2}{Z_j^2 + \|\mathbf{b}\|^2}. \end{aligned}$$

So,

$$\int_{Z_j>c} Z_j^2 \le \left(1+\frac{4\|\mathbf{b}\|^2}{c^2}\right)\int_{c<Z_j\le 2Y} Z_j^2 dP$$
$$\le \left(1+\frac{4\|\mathbf{b}\|^2}{c^2}\right)\int_{Y>\frac{1}{2}c} 2Y Z_j dP$$

and

$$\sum_{j\in J}\int_{Z_j>c} Z_j^2 \le 2\left(1+\frac{4\|\mathbf{b}\|^2}{c^2}\right)\int_{Y>\frac{1}{2}c} Y^2 dP$$

Apply to

$$Z_j = \pm\bar{b}_{n,j}\xi_{k,j} \quad \text{and} \quad Y = \pm D_{n,k}.$$