Homework 9

3 questions

Printed: April 3, 2020

Instructions. Solve the following problems. You can talk to other people about how to solve the exercises, but do not share written solutions. Be sure to state each exercise before solving it!

Notation: All random variables are defined on a probability space $(\Omega, \mathcal{F}, P)$. $X_n \xrightarrow{P} X$ denotes convergence in probability, $X_n \xrightarrow{\mathcal{D}} X$ denotes convergence in distribution.

Problem 1 (Exercise 9.2)**.** Prove that if $X_n \xrightarrow{P} c$ for a constant c and $Y_n \xrightarrow{\mathcal{D}} Y$, show that $X_n Y_n \xrightarrow{\mathcal{D}} cY$.

Solution:

Sol 1 $X_nY_n = cY_n + Y_n(X_n - c)$. Clearly $cX_n \xrightarrow{\mathcal{D}} cX$ by continuous mapping theorem, so the conclusion follows from Slutski's theorem, if we manage to show that $Y_n(X_n - c) \xrightarrow{P} 0$.

Fix two numbers $\varepsilon, K > 0$ such that K is a continuity point of $F_{|Y|}$. Then $P(|Y_n(X_n - c)| > \varepsilon) \le P(|Y_n| > K) + P(|X_n - c| > \varepsilon/K)$. So $\limsup_{n\to\infty} P(|X_n(Y_n - c)| > \varepsilon) \le \lim_{n\to\infty} P(|Y_n| > K) = P(|Y| > K) = 1 - F_{|Y|}(K)$. Here we used continuous mapping theorem: if $Y_n \xrightarrow{\mathcal{D}} Y$ then $|Y_n| \xrightarrow{\mathcal{D}} |Y|$. Since K is an arbitrary point of continuity and there is an uncountable number of continuity points in every interval, taking the limit as $K \to \infty$ ends the proof.

Sol 2 One can also use the definition directly. For this proof one need to consider separately the cases $c < 0, c = 0, c > 0$, so the details are omitted.

Problem 2 (Exercise 9.7)**.** Suppose $E(X_n^2) = 1$. Show that $\{X_n\}$ is uniformly integrable.

Solution: Use Cauchy-Schwartz and Markov: $\int_{|X_n|>K} |X_n| dP \le \sqrt{EX_n^2}\sqrt{P(|X_n| > K)} \le \frac{1}{K}$. So given $\varepsilon > 0$, take $K = 1/\varepsilon$. (There are other proofs)

Problem 3 (Exercise 9.12)**.** Suppose $(X_n, Y_n) \xrightarrow{\mathcal{D}} (X, Y)$. Prove that $X_n^2 + Y_n^2$ converges in distribution.

Solution: By Portmaneou theorem, it is enough to show that $E(f(X_n^2 + Y_n^2) \to E(f(X^2 + Y^2)$ for every bounded continuous function $f : \mathbb{R} \to \mathbb{R}$. But $f(x^2 + y^2) = f \circ g(x, y) = \varphi(x, y)$ defines a bounded continuous function $\varphi : \mathbb{R}^2 \to \mathbb{R}$ so convergence follow by our definition.