Homework 8

6 questions due when classes resume

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Instructions. Solve the following (*) problems. You can talk to other people about how to solve the exercises, but do not share written solutions. Be sure to state each exercise before solving it!

Notation: All random variables are defined on a probability space $(\Omega, \mathcal{F}, P)$. $X_n \xrightarrow{P} X$ denotes convergence in probability, $X_n \xrightarrow{\mathcal{D}} X$ denotes convergence in distribution.

Two prelim-like questions.

Problem 1 (*)**.** Suppose $\{X_n\}$ are independent with mean 0 and finite variances $Var(X_n) = \frac{n+1}{n}$. Let $S_n = \sum_{k=1}^{n} a_k X_1 X_2 \dots X_k$. Adapt the proof of Kolmogorov's maximal inequality to show that

$$P(\max_{1 \leq k \leq n} |S_k| \geq t) \leq \frac{\sum_{k=1}^{n}(k+1)a_k^2}{t^2}$$

Problem 2 (*)**.** $\boxed{\text{Problem 3 from Homework 7, corrected}}$ Suppose $X_1, X_2, \dots, X_n, \dots$ are independent exponential[1] random variables with parameters $\lambda_n \to \lambda > 0$. Modify the proof of the law of large numbers (which one?) to show that

$$\frac{1}{n}(X_1 + \cdots + X_n) \to 1/\lambda \text{ with probability one}$$

You can use "standard properties" of exponential random variables without proof. Hints: most proofs of SLLN can be adapted to this setting. But some details may vary or need adjustment.

[1] Recall that X_n is exponential with parameter $\lambda_n > 0$ if $P(X_n > t) = e^{-\lambda_n t}$ for $t \geq 0$

Solution: The limit value in this problem was mis-stated so as written the conclusion is false for all λ except $\lambda = 1$.

Here is the proof with correct limit and missing positivity assumption. If $\lambda_n \to \lambda > 0$ then

(i) $\frac{1}{\lambda_n}$ is a bounded sequence as it converges to $1/\lambda$.

(ii) $\frac{1}{n}\sum_{j=1}^n \frac{1}{\lambda_j} \to \frac{1}{\lambda}$ as averages of convergent sequences converge.

Denote $S_n = X_1 + \cdots + X_n$. Then $|\frac{1}{n}S_n - \frac{1}{\lambda}| \leq \frac{1}{n}(S_n - E(S_n))| + |\frac{1}{n}\sum_{j=1}^n \frac{1}{\lambda_j} - \lambda|$, so we only need to show that $\frac{1}{n}(S_n - E(S_n)) \to 0$ with probability one. This can be proved in many ways:

(a) Since $Var(S_n) = \sum_{k=1}^n \frac{1}{\lambda_k^2} \leq Mn$, we can use Borel-Cantelli to deduce that $\frac{1}{k^2}(S_k^2 - E(S_{k^2})) \to 0$ with probability one. So $\frac{1}{k^2}S_{k^2} \to \frac{1}{\lambda}$, and then use the usual trick $S_{(k-1)^2} \leq S_n \leq S_k^2$. (Note that to apply this technique we need to go back to the original un-centered sums!)

(b) One can compute a bound for the 4-th moments of the sum of independent centered random variables $X_k - E(X_k)$. In this approach, we get $\frac{1}{n}(S_n - E(S_n)) \to 0$ with probability one directly from Borel-Cantelli Lemma.

(c) One can use Kronecker's Lemma and Kolmogorov's one-series theorem:
$$\sum_{n=1}^{\infty} \frac{1}{n^2} Var(X_n) = \sum_{n=1}^{\infty} \frac{1}{n^2 \lambda_n^2} < \infty$$
so the series $\sum_{n=1}^{\infty} \frac{1}{n}(X_n - E(X_n))$ converges with probability one. This is probably the simplest proof for this question!!!

Lets compute exponential moments using extra parameter $t > 0$ to be chosen later:
$$P(|\frac{1}{n}(S_n - E(S_n))| > \varepsilon) = P(\exp t|S_n - E(S_n)| > e^{t\varepsilon n}) \leq e^{-t\varepsilon n} E \exp t|S_n - E(S_n)| \leq e^{-t\varepsilon n} E \exp t|S_n - \tilde{S}_n|$$
Now $e^{|x|} \leq e^x + e^{-x}$, so $E \exp t|S_n - E(S_n)| \leq E \exp t(S_n - E(S_n)) + E \exp t(E(S_n)) - S_n) = \prod_{k=1}^n E \exp t(X_k - E(X_k)) + \prod_{k=1}^n E \exp t(EX_k - X_k)$. This reduces the calculation to a single exponential random variable X with parameter λ:
$$Ee^{t(X - E(X))} = \lambda e^{-t/\lambda} \int_0^{\infty} e^{tx} e^{-\lambda x} dx = \frac{\lambda e^{-t/\lambda}}{\lambda - t}$$
and similarly $Ee^{-t(X-E(X))} = \frac{\lambda e^{t/\lambda}}{\lambda + t}$. To show that $\sum_n P(|\frac{1}{n}(S_n - E(S_n))| > \varepsilon) < \infty$, lets apply the root test. Noting that for non-negative sequences $\lim_{n\to\infty} \sqrt[n]{a_n + b_n} = \max\{\lim_{n\to\infty} \sqrt[n]{a_n}, \lim_{n\to\infty} \sqrt[n]{b_n}\}$ we have $\limsup_{n\to\infty} \log \sqrt[n]{P(|\frac{1}{n}(S_n - E(S_n))| > \varepsilon)} \leq -t\varepsilon - t/\lambda + \ln \frac{\lambda}{\lambda - t}$. Value $t = \varepsilon\lambda/(1 + \varepsilon\lambda)$ minimizes this expression, and we get
$$\limsup_{n\to\infty} \log \sqrt[n]{P(|\frac{1}{n}(S_n - E(S_n))| > \varepsilon)} \leq -\varepsilon + \ln(1 + \varepsilon\lambda) < 0$$
So almost sure convergence follows from Borel-Cantelli Lemma.

Convergence of series — turn in only 2 starred problems (*) (You may replace them with non-starred problems from the same group, if you really have to).

Problem 3 (*). Suppose $\{X_n\}$ are independent identically distributed with mean 0 and finite variance. Show that the series $\sum_{n=1}^{\infty} \frac{1}{n} X_n X_{n+1}$ converges almost surely.

Solution: Let $Y_n = \frac{1}{n}X_nX_{n+1}$. Then $E(Y_n) = 0$ and $E(Y_n^2) = \sigma^4/n^2$. Since $Y_1, Y_3, \dots, Y_{2n+2}, \dots$ are independent and $Y_2, Y_4, \dots, Y_{2n}, \dots$ are independent, by Theorem **??** both series $\sum_{k=0}^{\infty} Y_{2k+1}$ and $\sum_{k=1}^{\infty} Y_{2k}$ converge with probability one. If the first series converges on Ω_1 and the second converges on Ω_2, then the sum of both series converges on $(\Omega_1 \cap \Omega_2$, and this is the set of probability one: $P\Omega_1 \cap \Omega_2) = 1 - P(\Omega_1' \cup \Omega_2') \geq 1 - P(\Omega_1') - P(\Omega_2') = 1$.
Let $S_n = \sum_{k=1}^{n} Y_k$. The above argument shows that partial sums $\sum_{k=0}^{n}(Y_{2k+1} + Y_{2k+2}) = S_{2n+2}$ converge to some (random) limit S. To prove convergence of the series, we need to show more: we need to show that all partial sums S_n converge to the same limit. To complete the proof, we note that $S_n = S_{2m}$ or $S_n = S_{2m+1} = S_{2m} + Y_{2m+1}$. To complete the proof, we prove that $S_{2m+1} \to S$ by showing that $Y_{2n+1} \to 0$ with probability one. The latter follows from convergence of the series. Or from Borel-Cantelli: $P(|Y_n| > \varepsilon) \leq \frac{\sigma^4}{\varepsilon^2 n^2}$ so $\sum_n P(|Y_n| > \varepsilon) < \infty$. (Hence for any $\varepsilon > 0$ we have $P(|Y_n| > \varepsilon \text{ } i.o.) = 0$.)

Problem 4. Suppose X_n are independent identically distributed integrable with *symmetric distribution*: X_1 has the same law as $-X_1$. Prove that the series $\sum_n \frac{1}{n}X_n$ converges with probability one. Hint: Apply Kolmogorov's Three Series Theorem.

Solution: We verify condition (**??**) with $c = 1$.

$$\sum_n P\left(\frac{|X_n|}{n} > 1\right) = \sum_n P(|X_1| > n) \leq \int_0^{\infty} P(|X| > t)dt = E|X| < \infty$$

By symmetry, $E(X_n I_{|X_n| \leq n}) = 0$, so the second series converges. The verification of the last condition is based on Fubini's theorem as in the proof of Kolmogorov's SLLN. This proof uses the fact that $\sum_{n \geq x} \frac{1}{n^2} \sim C/x$ for $x > 1$, which is re-proved within the argument.

$$\sum_n \frac{1}{n^2} E(X_n^2 I_{|X_n| \leq n}) = E(\sum_n \frac{1}{n^2} X_n^2 I_{|X_n| \leq n})$$

$$= E\left(X^2 \sum_{n \geq |X|} \frac{1}{n^2}\right) \leq E(X^2 I_{|X| \leq 1}) + E\left(X^2 I_{|X|>1} \int_1^{|X|} \frac{du}{u^2}\right)$$

$$= E(X^2 I_{|X| \leq 1}) + E\left(X^2 I_{|X|<1}(\frac{1}{|X|} - 1)\right)$$

$$\leq E(X^2 I_{|X| \leq 1}) + E\left(X^2 \frac{1}{|X|}\right) \leq 1 + E|X| < \infty$$

Problem 5 (*). Suppose $Z_1, Z_2, \dots$ are independent normal random variables with $E(Z_n) = 0$ and $E(Z_n^2) = n$. For what $\theta > 0$ the series $\sum_{n=1}^{\infty} \frac{Z_n}{(n^2+n+1)^\theta}$ converges?

Problem 6. Suppose $X_1, X_2, \dots$ are independent random variables such that

$$P(X_n = n) = 1/(2n), \quad P(X_n = 0) = 1/2 - 1/n, \quad P(X_n = -1) = 1/(2n)$$

Use the three series theorem to determine for what values of θ the series $\sum_{n=1}^{\infty} \frac{X_n}{n^\theta}$ converges.

Convergence in distribution — turn in only 2 starred problems (*) (You may replace them with non-starred problems from the same group, if you really have to).

Problem 7. Suppose $P(X_n = k) = 1/n$ for $k = 1, 2, \dots, n$. Show that $\frac{1}{n}X_n \xrightarrow{\mathcal{D}} X$.

Solution: Indeed, $F_n(x) = [nx + 1]/n \to x$. Note however that $P(\frac{1}{n}X_n \in V)$ may fail to converge to $\lambda(V)$ for some Borel sets $V \in \mathcal{B}$.

Problem 8. Suppose $\{X_k\}$ are independent exponential. Show that $\max_{1 \leq k \leq n} X_k - \ln n \xrightarrow{\mathcal{D}} Y$ and determine the limiting distribution.

Solution: Indeed, $P(\max_{1\leq k\leq n} X_k - \ln n \leq x) = P(\max_{1\leq k\leq n} X_k \leq x + \ln n) = P(X_1 \leq x + \ln n)^n = (1 - e^{-x \ln n})^n = (1 - \frac{e^{-x}}{n})^n \to e^{-e^{-x}}$

Problem 9 (*). Suppose $\{X_k\}$ are independent uniform $U(0,1)$ random variables. Show that

$$n \min_{1\leq k\leq n} X_k \xrightarrow{\mathcal{D}} Y$$

and determine the law od Y.

Problem 10 (*). Suppose X_n has density $f_n(x) = 1 + \cos(2\pi n x)$ on $[0,1]$. Prove that $X_n \xrightarrow{\mathcal{D}} X$ (and determine the law of X).