Homework 6

not collected

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Instructions. You can talk to other people about how to solve the exercises, but do not share written solutions. Be sure to state each exercise before solving it!

Convergence in probability and in L_p.

Problem 1. Consider $\Omega = [0, 1)$ with the Lebesgue measure. Suppose

$$X_n(\omega) = \begin{cases} 2^n & \omega < 1/n \\ \omega & \omega \in [1/n, 1] \end{cases}$$

(i) Use the definition to show that $X_n \to X$ a.e. (Be sure to point out the exceptional set)

(ii) Show that $\|X_n - X\|_p \to \infty$ for any $p \geq 1$.

Problem 2. Suppose $\|X_n - X\|_4 \to 0$. Show that $X_n \xrightarrow{P} X$ and that $E(X_n) \to E(X)$.

Problem 3. If $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$, show that $X_n Y_n \xrightarrow{P} XY$.

Independence.

Problem 4. Suppose A, B, C are independent events. Recall that by a theorem in the book, sigma-fields $\sigma(A, B)$ and $\sigma(C)$ are independent. Prove directly from the definition that $(A \cup B), C$ is a pair of independent events.

Problem 5. Suppose X, Y are independent random variables, X is uniform $U(0, 1)$ with density $f(x) = 1_{(0,1)}(x)$ and Y is uniform on $\{0, 1, 2, 3, 4\}$ with $P(Y = j) = 1/5$, $j = 0, \dots, 4$ show that random variables $S = X + U$ and $T = 5X$ have the same distribution.

Problem 6. Consider $\Omega = (0, 1)$ with the Lebesgue measure. Define

$$X_n(\omega) = \begin{cases} 1 & \text{if } \lfloor 2^n \rfloor \text{ is even} \\ 0 \text{ if } \lfloor 2^n \rfloor \text{ is odd} \end{cases}$$

Show that X_1, X_2, X_3 are independent.

First Borel-Cantelli Lemma.

Problem 7 (Borel-Cantelli)**.** Suppose

$$P(|X_n| > t) \leq \frac{17}{t\sqrt{n}}$$

Determine an explicit subsequence n_k such that $X_{n_k} \to 0$ with probability.

Problem 8. Suppose $\bar{X}_n$ are square-integrable random variables with mean 17 and variance $17/n$. Show that $\bar{X}_{n^2} \to 17$ a.e.

Problem 9. Suppose

$$P(X_n = n^3) = \frac{1}{n^2} \quad P(X_n = 1) = 1 - \frac{1}{n^2}$$

Show that $X_n \to 1$ with probability one but $E(X_n)$ diverges.