Homework 4

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Instructions. You can talk to other people about how to solve the exercises, but do not share written solutions. Be sure to state each exercise before solving it!

Problem 1. Choose **one of the following**:

(i) Use Hölder's inequality to prove that

$$\int |f_1 f_2 f_3 f_4|\, d\mu \le \|f_1\|_4 \|f_2\|_4 \|f_3\|_4, \|f_4\|_4$$

(ii) Use Hölder's inequality to show that

$$\mathbb{E}(|XYZ|) \le \|X\|_3 \|Y\|_3 \|Z\|_3$$

Recall convergence in probability:

Definition 1. $X_n \xrightarrow{P} X$ if for every $\varepsilon > 0$ we have $\lim_{n\to\infty} P(|X_n - X| > \varepsilon) = 0$.

Problem 2 (L_p-convergence implies convergence in probability)**.** Show that if $\lim_{n\to\infty} \|X_n\|_p = 0$ for some $p \ge 1$, then $X_n \xrightarrow{P} 0$.

Hint: Markov!

Problem 3. Prove that if $X_n \xrightarrow{P} X$ and $Y_n \xrightarrow{P} Y$ then $X_n + Y_n \xrightarrow{P} X + Y$.

Hint: Use triangle inequality $a + b| \le |a| + |b|$ and ... *(a dose of ingenuity?)*

Problem 4. If $X_n \to X$ in probability and $X_n \ge 1$. a.e for all n, then $X \ge 1$ a.e. (that is, $P(X \ge 1) = 1$).

Hint: Try proof by contradiction or contrapositive.

Problem 5. If $X_n \xrightarrow{P} X$ in probability and $X_n \ge 1/100$ a.e. for all n, then $X \ge 0$ a.e. and $\sqrt{X_n} \xrightarrow{P} \sqrt{X}$ in probability.

Hint #1: Of course, we must have $X_n \ge 0$ to write $\sqrt{X_n}$. The assumption $X_n \ge 1/100$ is not really needed but may be helpful for some proofs.

Hint #2: How does one prove that $\sqrt{x}$ is continuous from definition? How does one compute the derivative of $\sqrt{x}$ from definition?

Additional exercises not to be turned in at this time

Problem 6. Suppose random variable X has cumulative distribution function

$$F(x) = \begin{cases} 0 & x < 0 \\ x/4 & 0 \le x < 1 \\ 1 & x \ge 1 \end{cases}$$

Compute $\mathbb{E}(X)$. *Hints:*

- *One way to do it is to construct a random variable on $\Omega = (0,1)$ and integrate the resulting function.*
- *Another way is to use change of variable formula $\int_{\mathbb{R}} x dF(x)$.*

Problem 7. Let $X \ge 0$ with $0 < \mathbb{E}X^2 < \infty$. Use Hölder's inequality to the product $XI\{X > 0\}$ to prove that $P(X > 0) \ge \frac{(\mathbb{E}(X))^2}{\mathbb{E}(X^2)}$.

Problem 8. Prove that if $X_n \xrightarrow{P} X$ and $a \in \mathbb{R}$ then $aX_n \xrightarrow{P} aX$.

Problem 9 (metric for convergence in probability)**.** Define $d(X,Y) = E\frac{|X-Y|}{1+|X-Y|}$.

(i) Prove that $X_n \xrightarrow{P} X$ iff $d(X_n, X) \to 0$. (Note that this requires two proofs!)

(ii) Prove that d satisfies triangle inequality $d(X,Z) \le d(X,Y) + d(Y,Z)$ so it is a "metric".