Homework 3

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Instructions. You can talk to other people about how to solve the exercises, but do not share written solutions. Be sure to state each exercise before solving it!

Comments:

- If it helps, you may restrict your solution only to the case of probability measures, $\mu(\Omega) = 1$
- If you do not feel like proving the formula from Problem 2, you can instead show how to use it to solve Problem 4.

Problem 1 (Exercise 1.4.1)**.** Show that if $f \geq 0$ and $\int f d\mu = 0$ then $f = 0$ a.e. *Hints:*

(i) Use the inequalities from Section 1.4 of the book.

(ii) Measure μ is continuous from above (Theorem 1.1.1) and

$$\{\omega : f(\omega) > 0\} = \bigcup_{\varepsilon \in \mathbb{Q}, \varepsilon > 0} \{\omega : f(\omega) \geq \varepsilon\} = \bigcup_{n \in \mathbb{N}} \{\omega : f(\omega) \geq \tfrac{1}{n}\}$$

Solution: We use the assumption $f \geq 0$ twice in the proof. First use: $f 1_{f \geq 1/n} \leq f$ so

$$\mu(f \geq 1/n) = \int 1_{f \geq 1/n} d\mu \leq \int n f 1_{f \geq 1/n} d\mu = n \int f 1_{f \geq 1/n} d\mu \leq n \int f d\mu = 0$$

and hence $\mu(f > 0) = \lim_{n \to \infty} \mu(f \geq 1/n) = 0$. Second use: $\mu(f < 0) = 0$, so $\mu(f \neq 0) = \mu(f > 0) + \mu(f < 0) = 0$

Problem 2 (Exercise 1.4.2)**.** For $f \geq 0$ and

(*) $$\Omega_{n,m} = \left\{\omega \in \Omega : \tfrac{m}{2^n} \leq f(\omega) < \tfrac{m+1}{2^n}\right\},$$

show that

$$\lim_{n \to \infty} \sum_{m=1}^{\infty} \tfrac{m}{2^n} \mu(\Omega_{n,m}) \nearrow \int f d\mu$$

Hint: Try using Lemma 1.4.4 with

$$E_n = \bigcup_{m=1}^{(n+1)2^n} \Omega_{n,m}$$

(For infinite μ, you'd need to consider cases! If you want to avoid them, assume $\mu(\Omega) = 1$ for this problem).

Solution: Lets solve this question using material from Section 1.5. Consider non-negative functions $g_n = \sum_{m=0}^{\infty} \frac{m}{2^n} 1_{\Omega_{m,n}}$. Then $0 \leq g_n \leq f \leq g_n + 1/2^n$ and $g_n \leq g_{n+1}$ so $g_n \nearrow f$.
By monotone convergence theorem, $\int g_n d\mu \nearrow \int f d\mu$. It remains to observe that $\int g_n d\mu = \lim_{M \to \infty} \int \sum_{m=0}^{M} \frac{m}{2^n} 1_{\Omega_{m,n}} d\mu = \sum_{m=0}^{\infty} \frac{m}{2^n} \mu(\Omega_{m,n})$.

Problem 3. On probability space $\Omega = (0, 1)$ with Borel σ-field and Lebesgue measure λ, compute $\mathbb{E}(X) := \int X(\omega) d\lambda$ for $X(\omega) = \min\{\omega, \frac{1}{2}\}$

(a) Using calculus/probability (with integration as in calculus).

(b) Using Exercise 1.4.2 on page 24 of the book. *Hint: In this case, $\Omega_{n,m}$ defined in* (*) *above is just an interval!*

Solution: For (a), $E(X) = \int_0^{1/2} x dx + \int_{1/2}^{1} 1/2 dx = \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$
For (b), we use sumation formula $0 + 1 + 2 + \cdots + M = M(M+1)/2$. We get $E(X) = \lim_{n \to \infty} \sum_{m=0}^{2^{n-1}} \frac{m}{2^n} \times \frac{1}{2^n} + \frac{1}{2} \times \frac{1}{2} = \frac{2^{n-1}(2^{n-1}+1)/2}{4^n} + \frac{1}{4} = \frac{4^n + 2 \times 2^n}{8 \times 4^n} + \frac{1}{4} \to \frac{1}{8} + \frac{1}{4} = \frac{3}{8}$

Optional

Problem 4. Use definition or Exercise 1.4.2 on page 24 of the book (not calculus!) to show that $\int_{[1,\infty)} \frac{1}{x}\,dx = \infty$ and $\int_{[1,\infty)} \frac{1}{x^2}\,dx < \infty$.

Hint: Notation $\int_{[1,\infty)} \frac{1}{x}\,dx$ *means* $\int f d\lambda$ *with respect to the Lebesgue measure* λ *on* $(\mathbb{R}, \mathcal{B})$ *for*

$$f(x) = \begin{cases} \frac{1}{x} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

(Similar notation $\int_1^\infty \frac{dx}{x}$ *would denote improper Riemann integral, and the answer would be the same, but we want to practice the Lebesgue integral here.)*

Solution: Comparing functions, we have $\frac{1}{x}1_{[1,\infty)}(x) \geq \sum_{n=1}^\infty \frac{1}{n+1}1_{[n,n+1)}(x)$. So the integrals compare: $\int_{[1,\infty)} \frac{1}{x}\,dx \geq \int_{[1,\infty)} \sum_{n=1}^\infty \frac{1}{n+1}1_{[n,n+1)}\,dx = \lim_{M\to\infty} \int_{[1,\infty)} \sum_{n=1}^M \frac{1}{n+1}1_{[n,n+1)}\,dx = \lim_{M\to\infty} \sum_{n=1}^M \frac{1}{n+1} = \infty$

For the second porblems, we compare $\frac{1}{x^2}1_{x\geq 1} \leq \sum_{n=1}^\infty \frac{1}{n^2}1_{[n,n+1)}(x)$ so $\int_{[1,\infty)} \frac{1}{x^2}\,dx \leq \sum_{n=1}^\infty \frac{1}{n^2} = \frac{\pi^2}{6} < 1 + \sum_{n=1}^\infty \frac{1}{n(n+1)} = 1 + \sum_{n=1}^\infty (\frac{1}{n} - \frac{1}{n+1}) = 2 < \infty$