Homework 2

Printed: January 27, 2020

Instructions. Turn in solution of one Stats problem and one Math problem You can talk to other people about how to solve the exercises, but do not share written solutions. Be sure to state each exercise before solving it!

Problem 1 (Stats)**.** If X has cdf F, and continuous density f, compute the cumulative distribution function of X^2 and differentiate to find the density. Apply this formula to standard normal distribution (Example 1.2.5) to determine the so called *chi-squared* density with one degree of freedom.

Solution:

$$F_{X^2}(x) = P(X^2 \leq x) = \begin{cases} 0 & x < 0 \\ \int_{-\sqrt{x}}^{0} f(y)dy + \int_0^{\sqrt{x}} f(y)dy & x > 0 \end{cases}$$

Since f iscontinuous, from calculus we know that this expression is differentiable at any $x < 0$ and at any $x > 0$, so the denisty g of X^2 is

$$g(x) = \begin{cases} 0 & x < 0 \\ 2020^3 & x = 0 \\ \frac{1}{2\sqrt{x}}\left(f(\sqrt{x}) + f(-\sqrt{x})\right) & x > 0 \end{cases}$$

When $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$ this formula (with the arbitrary value of 2020^3 replaced by 0 for convenience) gives

$$f(x) = \begin{cases} 0 & x \leq 0 \\ \frac{e^{-x/2}}{\sqrt{2\pi}\sqrt{x}} & x > 0 \end{cases}$$

Problem 2 (Stats)**.** On $\Omega = (0, 1)$, with Borel σ-field and Lebesgue measure, construct a random variable with cumulative distribution function

$$F(x) = \begin{cases} \frac{1+e^{-1/x}}{2} & x \geq 0 \\ \frac{e^x}{3} & x < 0 \end{cases}$$

Note: Although a good guess might work, I'd prefer a systematic solution that follows the method from the proof of Theorem 1.2.2 in the book.

Solution:

$$\{x \in \mathbb{R} : F(x) < \omega\} = \begin{cases} (-\infty, \ln(3\omega)) & \omega < 1/3 \\ (-\infty, 0) & \omega \in [\frac{1}{3}, \frac{1}{2}] \\ (-\infty, -1/\ln(2\omega - 1)) & \omega > 1/2 \end{cases}$$

So

$$X(\omega) = \begin{cases} \ln(3\omega) & \omega \leq 1/3 \\ 0 & 1/3 \leq \omega \leq 1/2 \\ \frac{-1}{\ln(2\omega-1)} & \omega > 1/2 \end{cases}$$

Problem 3 (Math)**.** On $\Omega = (-1, 1)$ with Borel σ-field $\mathcal{B}$, let $X(\omega) = \omega^2$, which of the following subsets of Ω are in $\sigma(X)$:

(i) $A_1 = (\frac{1}{4}, \frac{1}{2})$ $\not\subset \sigma(X)$. If $\omega \in X^{-1}(U)$ then $-\omega \in X^{-1}(U)$ so $\sigma(X)$ can only include symmetric intervals.

(ii) $A_2 = \{0\}$ $\subset \sigma(X)$ as $\{0\} = X^{-1}(\{0\})$

(iii) $A_3 = \{\frac{1}{2}\}$ $\not\subset \sigma(X)$ as $X^{-1}(\{\frac{1}{4}\}) = \{-\frac{1}{2}, \frac{1}{2}\}$

(iv) $A_4 = (-\frac{1}{4}, \frac{1}{4})$ $\subset \sigma(X)$ as $(-\frac{1}{4}, \frac{1}{4}) = X^{-1}([0, \frac{1}{16}))$

(v) $A_4 = (-\frac{1}{4}, \frac{1}{2})$ $\not\subset \sigma(X)$

(Give *some* justification, but no formal proof is required.)

Problem 4 (Math)**.** On $\Omega = (-1, 1)$ with Borel σ-field $\mathcal{B}$, consider the following random variables:

$$X_1(\omega) = \omega^2 \quad (1)$$
$$X_2(\omega) = \cos(\pi\omega) \quad (2)$$
$$X_3(\omega) = \sin(\pi\omega) \quad (3)$$
$$X_4(\omega) = \omega \quad (4)$$

Denote by $\sigma(X_j)$ their σ-fields. Decide which of the 6 pairs of σ-fields are different, which are the same. For those that are different give a set that is in one of them but not in the other. (Give *some* justification, but no formal proof is required.)

pairs	$\sigma(X_2)$	$\sigma(X_3)$	$\sigma(X_4)$
$\sigma(X_1)$	same	different	different
$\sigma(X_2)$	same	different	different
$\sigma(X_3)$		same	same

Solution: $\sigma(X_2) \subset \sigma(X_1)$ because $X_2(\omega) = \cos(\pi\omega) = \sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}\omega^{2n}}{(2n)!} = \sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}X_1^n}{(2n)!}$ is a continuous function of X_1.
$\sigma(X_1) \subset \sigma(X_2)$ because $X_1 = (\arccos(X_2)/\pi)^2$. This is because for $\alpha \in (-\pi, \pi)$ $\arccos\cos(\alpha) = |\alpha|$ and $|\omega|^2 = \omega^2$.
$\sigma(X_3) = \sigma(X_4) = \mathcal{B}((-1, 1))$, because both functions are strictly increasing on $(-1, 1)$, so any interval $(-1, a) \subset (\Omega)$ can be produced as an inverse image of an appropriate interval in $\mathbb{R}$, and intervals $(-1, a)$ generate $\mathcal{B}((-1, 1))$.
Clearly, $\sigma(X_1) \subset \mathcal{B} = \sigma(X_3) = \sigma(X_4)$ but $\sigma(X_1)$ is not equal to the entire $\mathcal{B}$. For example, Borel set $(0, 1)$ is not in $\sigma(X_1)$ because it cannot be written as $X_1^{-1}(U)$ as it lacks reflection symmetry. For any U, if $\omega \in X_1^{-1}(U)$ then $-\omega \in X_1^{-1}(U)$ and $(0, 1)$ does not have this symmetry.

Optional/saved for later use

Problem 5. Referring to Example 1.2.7, prove that Lebesque measure of the Cantor set $C \subset [0, 1]$ is 0.

Hint: $C = \bigcap_{n=1}^{\infty} C_n$ *with* $C_n = C_{n-1} \setminus U_n$ *where* $C_0 = [0, 1]$ *and in the* n*-th step we subtract the union* U_n *of (how many?) intervals of length* $\frac{1}{3^n}$

Problem 6. Use Exercise 1.4.2 on page 24 of the book (not calculus!) to show that $\int_{\mathbb{R}} x^2 \lambda(dx) = \infty$, $\int_{[1,\infty)} \frac{1}{x}\lambda(dx) = \infty$ and $\int_{[1,\infty)} \frac{1}{x^2}\lambda(dx) < \infty$.