Homework 10
Choose 2 questions of your interest

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Instructions. You can talk to other people about how to solve the exercises, but do not share written solutions. Be sure to state each exercise before solving it!

Notation: All random variables are defined on a probability space $(\Omega, \mathcal{F}, P)$. $X_n \xrightarrow{P} X$ denotes convergence in probability, $X_n \xrightarrow{\mathcal{D}} X$ denotes convergence in distribution.

Problem 1. Let X be a Poisson random variable with parameter λ. That is, $P(X = k) = e^{-\lambda}\lambda^k/k!$, $k = 0, 1, \ldots$. Compute the characteristic function $\varphi_\lambda(t)$ of $(X - \lambda)/\sqrt{\lambda}$ and find its limit as $\lambda \to \infty$. *Hint: Recall Taylor polynomial bounds*

$$|e^{ix} - 1| \leq \min\{|x|, 2\} \tag{1}$$

$$|e^{ix} - (1 + ix)| \leq \min\{\frac{1}{2}x^2, 2|x|\} \tag{2}$$

$$|e^{ix} - (1 + ix - \frac{1}{2}x^2)| \leq \min\{\frac{1}{6}|x|^3, x^2\} \tag{3}$$

Problem 2. Suppose $X_1, X_2, \ldots$ are i.i.d. with $P(X = \pm 1) = 1/2$. Let $S_n = X_1 + \cdots + X_n$. Compute the characteristic function of $\frac{1}{\sqrt{n}}S_n$ and find its limit as $n \to \infty$. *Hint: L'Hospital rule should be applicable.*

Problem 3. Suppose U_n are uniform on $(-n, n)$. Compute the characteristic function $\varphi_n(t)$ and find its limit as $n \to \infty$.

Problem 4. Suppose X, Y are independent exponential (i.e. with density e^{-x} for $x > 0$). Compute the characteristic function of $X - Y$.

Problem 5. Suppose $X_n \xrightarrow{\mathcal{D}} X$ and $a_n \to a$, $b_n \to b$. Use characteristic functions to show that $a_n X_n + b_n \to aX + b$.

Problem 6. Suppose $X_n \xrightarrow{\mathcal{D}} X$, $Y_n \xrightarrow{\mathcal{D}} Y$ and each pair (X_n, Y_n) consists of independent random variables (on some probability space Ω_n). Show that $X_n + Y_n \xrightarrow{\mathcal{D}} S$ where the law of S is the convolution of the laws of X and Y.

Problem 7. Show that

$$P(X = a) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-ita}\varphi(t)dt$$