1. Tail integration add-on from 2019 Notes

1.1. Tail integration formula. If $X \geq 0$ then

$$E(X) = \int_0^\infty P(X > x)dx = \int_0^\infty P(X \geq x)dx \tag{T}$$

First Proof. For simple random variables this is just a picture. For general X, take simple $X_n \nearrow X$. Noting that $I_{X_n > t} \nearrow I_{X>t}$ we get $P(X_n > t) \nearrow P(X > t)$, the result follows from the monotone convergence theorem applied to $f_n(t) = P(X_n > t)$. □

Second Proof. Formula (T) and its various generalizations are easy to derive from Fubini's theorem. Lets get a more general version of the formula. If $X \geq 0$ and $p \geq 1$ then $X^p = \int_0^X pt^{p-1}dt = \int_0^\infty pt^{p-1}I_{t<X}dt$ so

$$E(X^p) = \int_\Omega \int pt^{p-1}I_{t<X}dtdP(\omega) = \int_0^\infty \int_\Omega pt^{p-1}I_{t<X}dP(\omega)dt = p\int_0^\infty t^{p-1}P(X > t)dt \tag{T+}$$

This formula holds true also in the non-integrable case - both sides are then ∞. Formula (T) is of course case $p = 1$ of (T+). □

Homework 5

Turn in one discrete problem D and one continuous problem C.

Problem 1 (D). Suppose that X is a simple random variable which has non-negative integers $\{0, 1, 2, \dots\}$ as values. Use (T), or some other means, to prove that

$$\mathbb{E}(X^2) = \sum_{n=1}^{\infty}(2n-1)P(X \geq n)$$

Solution: Lets use tail integration formula: $\mathbb{E}\left[X^2\right] = \int_0^\infty P(X^2 > t)dt = \int_0^\infty P(X > \sqrt{t})dt = \int_0^\infty 2uP(X > u)du = \sum_{n=0}^\infty \int_n^{n+1} 2uP(X > u)du = \sum_{n=0}^\infty \int_n^{n+1} 2uP(X \geq n+1)du = \sum_{n=0}^\infty ((n+1)^2 - n^2)P(X \geq n+1) = \sum_{n=0}^\infty (2n+1)P(X \geq n+1) = \sum_{n=1}^\infty (2n-1)P(X \geq n)$

Problem 2 (D). Suppose that X is a simple random variable which has non-negative integers $\{0, 1, 2, \dots\}$ as values. Prove that

$$\mathbb{E}\left[2^X\right] = 1 + \sum_{n=0}^{\infty} 2^n P(X \geq n+1)$$

Solution: Lets proceed with the right hand side: $\sum_{n=0}^\infty 2^n P(X \geq n+1) = \sum_{n=0}^\infty 2^n \sum_{k=n+1}^\infty P(X = k) = \sum_{k=1}^\infty P(X = k)\sum_{n=0}^{k-1} 2^n = \sum_{k=1}^\infty P(X = k)(2^k - 1) = \sum_{k=1}^\infty 2^k P(X = k) - \sum_{k=1}^\infty P(X = k) = \sum_{k=1}^\infty 2^k P(X = k) - (1 - p_0) = \sum_{k=0}^\infty 2^k P(X = k) - 1 = \mathbb{E}[X] - 1$

Problem 3 (D). Suppose that X is a simple random variable which has positive integers $\{1, 2, \dots\}$ as values. Use (T), to prove that

$$\mathbb{E}\left[\frac{1}{X}\right] = 1 - \sum_{n=1}^{\infty} \frac{1}{n(n+1)} P(X \geq n+1)$$

Solution: Here one can proceed either with tail integration and change of variable as in Exercise 3, or work with the right hand side as in Exercise 2.
First method requires noticing that $P(1/X > t) = 0$ when $t > 1$, and then working out $\mathbb{E}[X] = \int_0^\infty P(1/X > t)dt = \int_0^1 P(X < 1/t)dt = 1 - \int_0^1 P(X > 1/t)dt = 1 + \int_1^\infty 1/u^2 P(X > u)du = 1 - \sum_{n=1}^\infty \int_n^{n+1} 1/u^2 P(X > u)du = 1 - \sum_{n=1}^\infty P(X \geq n+1) \int_n^{n+1} 1/u^2 du = 1 - \sum_{n=1}^\infty P(X \geq n + 1) \int_n^{n+1} 1/u^2 du...$
The second method requires noticing the telescoping sum: $\sum_{n=1}^\infty \frac{1}{n(n+1)} P(X \geq n+1) = \sum_{n=1}^\infty (\frac{1}{n} - \frac{1}{n+1}) \sum_{k=n+1}^\infty P(X = k) = \sum_{k=2}^\infty P(X = k) \sum_{n=1}^{k-1} (\frac{1}{n} - \frac{1}{n+1}) = \sum_{k=2}^\infty P(X = k)(1 - \frac{1}{k}) = (1 - p_1) - \sum_{k=2}^\infty \frac{1}{k} P(X = k) 1 - \sum_{k=1}^\infty \frac{1}{k} P(X = k) = 1 - \mathbb{E}[1/X]$

Problem 4 (D). Use (T) to compute $\mathbb{E}(X)$ if $P(X = k) = (1 - q)q^{k-1}$, $k = 1, 2, \ldots$ for $0 < q < 1$. *Hint: Geometric series formula is* $\sum_{k=n}^\infty x^k = \frac{x^n}{1-x}$.

Problem 5 (C). Use Fubini's theorem (not tail integration formula (T)) to show that if $X \geq 0$ then

$$E\frac{1}{1+X} = 1 - \int_0^\infty \frac{1}{(t+1)^2} P(X > t)dt$$

Then re-derive the same result from (T).

Problem 6 (C). Use Fubini's theorem (not tail integration formula (T)) to show that if $X \geq 0$ then

$$Ee^X = 1 + \int_0^\infty e^t P(X > t)dt$$

Then re-derive the same result from (T).

Solution: Lets prove a more general fact: if $f(x) = f(0) + \int_0^x g(u)du$ and $g \geq 0$ then

$$\mathbb{E}(f(X)) = f(0) + \int_0^\infty g(t)P(X > t)dt$$

Proof. $f(X(\omega)) = f(0) + \int_0^\infty 1_{u<X(\omega)} g(u)du$ Since $g \geq 0$ we can apply Fubini's theorem:

$$\begin{aligned}\mathbb{E}(f(X) = \int_\Omega f(X(\omega))dP(\omega) &= \int_\Omega \left(f(0) + \int_0^\infty 1_{u<X(\omega)} g(u)du\right) dP(\omega) \\ &= f(0) + \int_0^\infty \int_\Omega 1_{u<X(\omega)} dP(\omega) g(u)du \\ &= f(0) + \int_0^\infty P(u < X)g(u)du\end{aligned}$$

□

Problem 7 (C). Suppose that $X, Y \geq 0$ are possibly dependent random variables and $p, q > 0$. Prove that

$$E(X^p Y^q) = pq \int_0^\infty \int_0^\infty t^{p-1} s^{q-1} P(X > t, Y > s)dtds$$

Integrability from Tail.

Problem 8. Suppose $\lim_{n\to\infty} n^2 P(|X| > n) < \infty$. Prove that $\mathbb{E}|X| < \infty$. *Hint: Use* (T)

Problem 9. Suppose $P(|X| > n) \leq 1/2^n$ for all n. Prove that there exists $\delta > 0$ such that $\mathbb{E}(e^{\delta|X|}) < \infty$. *Hint: Use Problem 6 (or* (T)*).*

Solution: We want to prove that $\int_0^\infty e^{t\delta} P(|X| > t)dt < \infty$. By integral test for convergence, this is equivalent with $\sum_{n=1}^\infty e^{\delta n} P(|X| > n) < \infty$. From the information provided, this series converges for any $\delta < \ln 2$.