1. Tail integration add-on from 2019 Notes

1.1. Tail integration formula. If $X \geq 0$ then

$$E(X) = \int_0^\infty P(X > x)dx = \int_0^\infty P(X \geq x)dx \tag{T}$$

First Proof. For simple random variables this is just a picture. For general X, take simple $X_n \nearrow X$. Noting that $I_{X_n > t} \nearrow I_{X>t}$ we get $P(X_n > t) \nearrow P(X > t)$, the result follows from the monotone convergence theorem applied to $f_n(t) = P(X_n > t)$. □

Second Proof. Formula (T) and its various generalizations are easy to derive from Fubini's theorem. Lets get a more general version of the formula. If $X \geq 0$ and $p \geq 1$ then $X^p = \int_0^X pt^{p-1}dt = \int_0^\infty pt^{p-1}I_{t<X}dt$ so

$$E(X^p) = \int_\Omega \int pt^{p-1}I_{t<X}dtdP(\omega) = \int_0^\infty \int_\Omega pt^{p-1}I_{t<X}dP(\omega)dt = p\int_0^\infty t^{p-1}P(X > t)dt \tag{T+}$$

This formula holds true also in the non-integrable case - both sides are then ∞. Formula (T) is of course case $p = 1$ of (T+). □