1. Integration add-on from 2019 Notes

Of special interest are cumulative distribution functions such that $F(x) = \int_{-\infty}^{x} f(y)dy$ where $f(y) \geq 0$ is called the *density function*, i.e. a non-negative measurable and integrable function that integrates to 1. (We do not assume continuity! Improper Riemann integrals are OK here.)

Proposition 1.0.1. *If random variable X is integrable and has cumulative distribution function $F(x) = \int_{-\infty}^{x} f(y)dy$ then*

$$\mathbb{E}[X] = \int_{\mathbb{R}} x f(x) dx. \tag{1}$$

To prove this, we consider separately X^+ and X^-.

We decompose $X = X^+ - X^-$ and approximate $\psi_n(X^+) \uparrow X^+$ with $\psi_n(x) = (k-1)/2^n$ on $((k-1)/2^n, k/2^n]$, $\psi_n(x) = 0$ for $x < 0$, compare (**??**). Then

$$E(\psi_n(X^+)) = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \left(F(\frac{k}{2^n}) - F(\frac{k-1}{2^n}) \right) = \int_{\mathbb{R}} \psi_n(x) f(x) dx.$$

Taking the limit, by monotone convergence theorem (see Remark **??**) we get $E(X^+) = \int_{\mathbb{R}} x^+ f(x) dx$ and hence $E(X) = \int_{\mathbb{R}} (x^+ - x^-) f(x) dx = \int_{\mathbb{R}} x f(x) dx$

Example 1.0.2. *Uniform density* $U[a,b]$ is $f(x) = \frac{1}{b-a} I_{[a,b]}(x)$. The mean and the variance are $m = (a+b)/2$, $\sigma^2 = (b-1)^2/12$

Example 1.0.3. *Exponential distribution*: $F(x) = \begin{cases} 1 - e^{-x} & x > 0 \\ 0 & x \leq 0 \end{cases}$. The density is $f(x) = e^{-x} I_{[0,\infty)}(x)$. The mean and the variance are $m = 1$, $\sigma^2 = 1$.

Example 1.0.4. *Standard normal density*: $f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$. The mean and the variance are $m = 0$ and $\sigma^2 = 1$.

1.0.1. *Multivariate densities.* Similar approximation argument shows that if $\mu(d\mathbf{x}) = f(\mathbf{x})d\mathbf{x}$ has the density with respect to Lebesgue measure on $\mathbb{R}^k$ then

$$\mathbb{E}[g(X_1, \dots, X_k)] = \int_{\mathbb{R}^k} g(\mathbf{x}) f(\mathbf{x}) d\mathbf{x}$$

In particular, $cov(X_1, X_2) = \int_{\mathbb{R}^2} (x - m_1)(y - m_2) f(x_1, x_2) dx_1 dx_2$

Example 1.0.5. Uniform distribution on a disk: $f(x,y) = \begin{cases} \frac{1}{\pi} & x^2 + y^2 \leq 1 \\ 0 & x^2 + y^2 > 1 \end{cases}$