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(1) Suppose that random variables $X_1, X_2, \dots$ are i.i.d. with $E(X_j) = 0$ and $E(X_j^4) = 17$. Let $S_n = X_1 + \cdots + X_n$. Prove that

$$\frac{1}{n^{7/8}} S_n \to 0 \text{ with probability one.}$$

(2) State and prove Kolmogorov's maximal inequality.

(3) State and prove Slutski's theorem.

(4) Suppose $\{X_k\}$ are independent uniform $U(0,1)$ random variables. Show that

$$n \min_{1 \le k \le n} X_k \xrightarrow{\mathcal{D}} Y$$

and determine the law od Y.

(5) Suppose $X_n \xrightarrow{\mathcal{D}} X$ and $Y_n > 0$ are such that $Y_n \xrightarrow{P} c$. Prove that $\frac{X_n}{Y_n} \xrightarrow{\mathcal{D}} \frac{1}{c} X$.

(6) Suppose $X_1, X_2, \dots$ are independent identically distributed uniform $U(1, e)$. Let $Z_n = X_1 \dots X_n$. Show that $\sqrt[n]{Z_n}$ converges almost surely (and find the limit).

(7) Suppose $X_1, X_2, \dots,$ are i.i.d. with mean m and variance $\sigma^2 > 0$. Let $\bar{X}_n = \frac{1}{n} \sum_{k=1}^n X_k$ be the sample mean and $S_n^2 = \frac{1}{n-1} \sum_{k=1}^n (X_k - \bar{X}_n)^2$ be the sample variance. Show that with probability one $S_n \to \sigma$.

(8) Suppose $X_1, X_2, \dots,$ are i.i.d. with mean m and variance $\sigma^2 > 0$. Show that with probability one

$$L = \lim_{n \to \infty} \frac{\sum_{k=1}^n X_k}{\sum_{k=1}^n X_k^2}$$

exists, and determine L as a function of m, σ.

(9) Suppose that random variables $X_1, X_2, \dots$ are independent identically distributed, with mean m, variance σ^2, and with finite 4-th moments. Let $S_n = \sum_{k=1}^n X_k X_{k+1}$. Show that $\frac{1}{n} S_n$ converges almost surely (and find the limit).

(10) Suppose $X_1, X_2, \dots$ are independent, square-integrable, with mean zero. Define $Y_n = X_1 X_2 \dots X_n$ and $S_n = Y_1 + \cdots + Y_n$. Adapt the proof of Kolmogorov's maximal inequality to prove that

$$P(\max_{k \le n} |S_k| > t) \le \frac{Var(S_n)}{t^2}$$

(Note that $Y_1, Y_2, \dots$ are dependent!)

(11) Suppose that random variables $\{X_k\}$ are independent uniform $U(\frac{1}{k} - k, k - \frac{1}{k})$ for $k \ge 2$. Show that the series $\sum_{n=2}^\infty \frac{1}{(n^3 - 2n + \frac{3}{n^3})^\theta} X_n$ converges with probability one for $\theta > \theta_0$, where θ_0 is to be determined by you.

(12) Let $X_1, X_2, \ldots$ be independent random variables with the distribution $(k \geq 1)$

$$\begin{aligned} \Pr(X_k &= \pm\frac{1}{k}) = 1/4, \\ \Pr(X_k &= k^k) = 1/4^k, \\ \Pr(X_k &= 0) = 1/2 - 1/4^k. \end{aligned}$$

Use Kolmogorov's three-series theorem to prove that the series $\sum k^{1/3} X_k$ converges with probability one.

(13) Suppose random variables Z_n with laws μ_n are such that $E(Z_n^2) = 1$. Show that $\{\mu_n\}$ is tight.