Printed February 29, 2020 Name ______________________________

Probability Exam 1 2020

Instructions. Write your proofs on separate sheets of paper.Turn in this page with your work.

Notation. All random variables are denoted by X, Y, Z (with subscripts) and are defined on the same probability space $(\Omega, \mathcal{F}, P)$. Abbreviations "a. e." for "almost everywhere" or "a. s." for "almost surely" mean the same as "with probability one". Notation $X_n \xrightarrow{P} X$ denotes convergence in probability as $n \to \infty$.

Questions.

(1) State and prove one of the Borel-Cantelli Lemmas. (Your choice which one) Proofs of these theorems are in every book on probability theory. The starting point for both proofs is continuity of probability measures: Given a sequence $\{A_k\}$ of events in $\mathcal{F}$, events $B_n = \bigcup_{k=n}^{\infty} A_k$ are monotone, $B_{n+1} \subset B_n$. So $P(\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty} A_k) = P(\bigcap_{n=1}^{\infty} B_n) = \lim_{n\to\infty} P(B_n) = \lim_{n\to\infty} P(\bigcup_{k=n}^{\infty} A_k)$

(2) Suppose that X, Y are square-integrable random variables. State and prove the Cauchy-Schwarz inequality. (Recall that this is Hl̈der's inequality with $p = q = 2$. If you prefer, you may choose to prove the general Hölder's inequality.) Two proofs of $(E(XY))^2 \le (E(|XY|))^2 \le E(X^2)E(Y^2)$ are in the book.

(3) For a non-negative random variable X, show that

$$E(X^2) = 2\int_0^\infty t\, P(X > t)\, dt.$$

A more general formula is proved in the book by the same method. Let's apply Fubini's theorem to non-negative function $2t1_{X(\omega)>t}$ (for $t \ge 0$):

$$E(X^2) = \int_\Omega \left(\int_0^{X(\omega)} 2t\, dt\right) dP = \int_\Omega \left(\int_0^\infty 1_{X(\omega)>t} 2t\, dt\right) dP = \text{(Fubini)}$$

$$= \int_0^\infty \left(\int_\Omega 1_{X(\omega)>t} 2t\, dP\right) dt = 2\int_0^\infty t \int_\Omega 1_{X(\omega)>t}\, dP\, dt = 2\int_0^\infty t\, P(X > t)\, dt.$$

(4) Suppose that random variables $S_1, S_2, \dots$ have mean zero and variances $E(S_n^2) = n^3$.

(a) Show that $\frac{1}{n^2} S_n \xrightarrow{P} 0$. Fix $\varepsilon > 0$. By Markov inequality we have $P(|\frac{1}{n^2} S_n| > \varepsilon) = P(|S_n| > \varepsilon n^2) \le \frac{E(S_n^2)}{\varepsilon^2 n^4} = \frac{1}{\varepsilon^2 n} \to 0$

(b) Determine an explicit sequence n_k such that $\frac{1}{n_k^2} S_{n_k} \to 0$ a.e. (Proof of convergence is required.) Take $n_k = k^2$. Then by the previous inequality $P(|\frac{1}{n_k^2} S_{n_k}| > \varepsilon) \le \frac{1}{\varepsilon^2 n_k} = \frac{1}{\varepsilon^2 k^2}$. So for every $\varepsilon > 0$, $\sum_{k=1}^{\infty} P(|\frac{1}{n_k^2} S_{n_k}| > \varepsilon) < \infty$. By First Borel Cantelli Lemma, $P(|\frac{1}{n_k^2} S_{n_k}| > \varepsilon \text{ i. o.}) = 0$ so for all rational $\varepsilon > 0$, $P(\frac{1}{n_k^2} S_{n_k}| \le \varepsilon \text{ finitely often}) = 1$. Since countable intersection of sets of probability 1 has probability one, $P(\forall_{\varepsilon\in\mathbb{Q}_+}\exists_n\forall_{k>n}\ \frac{1}{n_k^2}|S_{n_k}| \le \varepsilon) = 1$. (Notice that in the definition of convergence of sequences of real numbers we do not need all $\varepsilon > 0$ – it is enough to take $\varepsilon = 1/m$ for some $m \in \mathbb{N}$)

(5) If $X_n \xrightarrow{P} X$, show that $2X_n + \frac{n}{n+1} Y \xrightarrow{P} 2X + Y$ for any random variable Y. Since $\frac{n}{n+1} = 1 - \frac{1}{n+1} \to 1$, we have $\frac{n}{n+1} Y(\omega) \to Y(\omega)$ for every $\omega \in \Omega$. Convergence almost everywhere implies convergence in probability, so $\frac{n}{n+1} Y \xrightarrow{P} Y$. Fix $\varepsilon > 0$. Then by triangle inequality

$$P\left(\left|2X_n + \tfrac{n}{n+1} Y - (2X + Y)\right| > \varepsilon\right) \le P(2|X_n - X| + |\tfrac{n}{n+1} Y - Y)| > \varepsilon)$$

(Since $a + b > \varepsilon$ implies $\{a > \varepsilon/2\} \vee \{b > \varepsilon/2\}$)

$$\le P(|X_n - X| > \varepsilon/4) + P(|\tfrac{n}{n+1} Y - Y)| > \varepsilon/2) \to 0$$