Printed April 22, 2020 Name ______________________________

PROBABILITY STAT 7032 FINAL – 2020 EDITION

Instructions. This is a take-home final, due **Saturday, May 2, NOON**.

It has two components.

(1) Turn in solutions for **two Exercises** of your choice from Exercises 11.1-11.7 in the notes
(2) Turn in solution of one of the projects below. In lieu of the project, you can solve a CLT problem from one of the old exams. (Some problems are listed at the end of Ch. 11 of the notes.)

List of Projects.

Project 1. *Suppose X_k are independent with the distribution*

$$X_k = \begin{cases} 1 & \text{with probability } 1/2 - p_k \\ -1 & \text{with probability } 1/2 - p_k \\ k^\theta & \text{with probability } p_k \\ -k^\theta & \text{with probability } p_k \end{cases}$$

and $S_n = \sum_{k=1}^n X_k$. It is "clear" that if $\sum p_k < \infty$ then $S_n/\sqrt{n} \xrightarrow{\mathcal{D}} N(0, \sigma^2)$ for any θ. It is "clear" that if $\theta = 0$ then $S_n/\sqrt{n} \xrightarrow{\mathcal{D}} N(0, 1)$ for any choice of $p_k < 1/2$.

So it is natural to ask what assumptions on θ and p_k will imply asymptotic normality. In paricular,

- *What are the "optimal" restrictions on p_k if $\theta < 0$? (Say, if $\theta = -1$, to ease the calculations)*
- *Can one "do better" than $\sum p_k < \infty$ if $\theta > 0$? (Say, if $\theta = 1$, to ease the calculations)*

Project 2. *Suppose ξ_k are i.i.d. with mean zero and variance 1. Do "geometric moving averages"*

$$X_k = \sum_{j=0}^{k} q^j \xi_{k-j}$$

satisfy the CLT when $|q| < 1$?

That is, with $S_n = \sum_{k=1}^n X_k$ do we have $(S_n - a_n)/b_n \xrightarrow{\mathcal{D}} N(0, 1)$ for appropriate normalizing constants a_n, b_n? And if so, how does b_n depend on the q?

Project 3. *Suppose X_k are i.i.d. with density $\frac{1}{|x|^3}$ for $|x| > 1$. Show that $\frac{S_n}{\sqrt{n \log n}} \to N(0, 1)$ using one of the* **other truncations** *from the hint for Exercise 11.5 in the notes.*

Project 4. *Suppose Z_1, Z_2, Z_3, Z_4 be independent normal random variables. Let $\mathbf{Z}_\mathbb{C} = Z_1 + iZ_2$ be a complect random variable and $\mathbf{Z}_\mathbb{Q} = Z_1 + iZ_2 + jZ_3 + kZ_4$ be a quaternionic random variable.*

- *Show that*

$$\mathbb{E} Z_1^n = \begin{cases} \frac{n!}{2^{n/2}(n/2)!} & \text{if } n \text{ is even} \\ 0 & \end{cases}$$

- *What is the formula for $\mathbb{E}(\mathbf{Z}_\mathbb{C}^n)$ and for $\mathbb{E}(\mathbf{Z}_\mathbb{C}^m \bar{\mathbf{Z}}_\mathbb{C}^n)$ for $m, n = 0, 1, 2, \dots$?*
- *What is the formula for $\mathbb{E}(\mathbf{Z}_\mathbb{Q}^n)$ and for $\mathbb{E}(\mathbf{Z}_\mathbb{Q}^m \bar{\mathbf{Z}}_\mathbb{Q}^n)$ for $m, n = 0, 1, 2, \dots$*
- *Why don't I dare to ask about this formula for octonions?*

Project 5. *A multinomial experiment has k outcomes with probabilities $p_1, \dots, p_k$. A multinomial random variable $(N_1, \dots, N_k) = (N_1^{(n)}, \dots, N_k^{(n)})$ lists the observed counts per category in n repeats of the multinomial experiment. Let $E_j = E_j^{(n)} = np_j$ denote the expected counts in j-th category. Use the multivariate CLT to prove that*

$$\sum_{j=1}^{k} \frac{(N_j - E_j)^2}{E_j} \xrightarrow{\mathcal{D}} \chi^2_{k-1} \text{ as } n \to \infty$$

where $\chi^2_{k-1} := Z_1^2 + \dots + Z_{k-1}^2$ for some i.i.d. N(0,1) random variables.

Project 6. *Suppose ξ_j, η_j, γ_j are i.i.d. with mean m and variance σ^2. Construct the following vectors:*

$$\mathbf{X}_j = \begin{bmatrix} \xi_j - \eta_j \\ \eta_j - \gamma_j \\ \gamma_j - \xi_j \end{bmatrix}$$

Let $\mathbf{S}_n = \mathbf{X}_1 + \cdots + \mathbf{X}_n$. Show that $\frac{1}{n}\|\mathbf{S}_n\|^2 \xrightarrow{\mathcal{D}} Y$, and determine the density of Y.

Project 7. *Select a CLT problems from one of the past prelims. Turn in the solution.*