Two.III Basis and Dimension

Linear Algebra

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Basis

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Because a basis is a sequence, meaning that bases are different if they contain the same elements but in different orders, we denote it with angle brackets $\langle \vec{\beta}_1, \vec{\beta}_2, \ldots \rangle$.

Example This is a basis for $\mathbb{R}^2$.

$$\langle \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix} \rangle$$

It is linearly independent.

$$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{matrix} c_1 + c_2 = 0 \\ -c_1 + c_2 = 0 \end{matrix} \implies c_1 = 0, \, c_2 = 0$$

And it spans $\mathbb{R}^2$ since

$$c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} \implies \begin{matrix} c_1 + c_2 = x \\ -c_1 + c_2 = y \end{matrix}$$

has the solution $c_1 = (1/2)x - (1/2)y$ and $c_2 = (1/2)x + (1/2)y$.

Example In the vector space of linear polynomials $\mathcal{P}_1 = \{a + bx \mid a, b \in \mathbb{R}\}$ one basis is $B = \langle 1 + x, 1 - x \rangle$.

Check that is a basis by verifying that it is linearly independent

$$0 = c_1(1 + x) + c_2(1 - x) \implies 0 = c_1 + c_2, \; 0 = c_1 - c_2 \implies c_1 = c_2 = 0$$

and that it spans the space.

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Example This is a basis for $\mathcal{M}_{2\times 2}$.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \rangle$$

This is another one.

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Example This is a basis for $\mathbb{R}^3$.

$$\mathcal{E}_3 = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

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1.5 *Definition*   For any $\mathbb{R}^n$

$$\mathcal{E}_n = \langle \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} \rangle$$

is the *standard* (or *natural*) basis. We denote these vectors $\vec{e}_1, \ldots, \vec{e}_n$.

Checking that $\mathcal{E}_n$ is a basis for $\mathbb{R}^n$ is routine.

Although a basis is a sequence we will follow the common practice and refer to it as a set.

1.12 *Theorem*   In any vector space, a subset is a basis if and only if each vector in the space can be expressed as a linear combination of elements of the subset in one and only one way.

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1.12 *Theorem*   In any vector space, a subset is a basis if and only if each vector in the space can be expressed as a linear combination of elements of the subset in one and only one way.

Proof   A sequence is a basis if and only if its vectors form a set that spans and that is linearly independent. A subset is a spanning set if and only if each vector in the space is a linear combination of elements of that subset in at least one way. Thus we need only show that a spanning subset is linearly independent if and only if every vector in the space is a linear combination of elements from the subset in at most one way.

Consider two expressions of a vector as a linear combination of the members of the subset. Rearrange the two sums, and if necessary add some $0 \cdot \vec{\beta}_i$ terms, so that the two sums combine the same $\vec{\beta}$'s in the same order: $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$ and $\vec{v} = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + \cdots + d_n\vec{\beta}_n$. Now

$$c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n = d_1\vec{\beta}_1 + d_2\vec{\beta}_2 + \cdots + d_n\vec{\beta}_n$$

holds if and only if

$$(c_1 - d_1)\vec{\beta}_1 + \cdots + (c_n - d_n)\vec{\beta}_n = \vec{0}$$

holds. So, asserting that each coefficient in the lower equation is zero is the same thing as asserting that $c_i = d_i$ for each i, that is, that every vector is expressible as a linear combination of the $\vec{\beta}$'s in a unique way.   QED

Example Consider the vector space $\mathbb{R}^3$ and this vector and basis.

$$\vec{v} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \in \mathbb{R}^3 \qquad B = \langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rangle = \langle \vec{\beta}_1, \vec{\beta}_2, \vec{\beta}_3 \rangle$$

Find how to express $\vec{v}$ as $c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + c_3\vec{\beta}_3$ by solving this system.

$$\begin{aligned} c_1 + c_2 + c_3 &= 1 \\ c_1 + c_2 \phantom{{}+ c_3} &= 2 \\ c_1 \phantom{{}+ c_2 + c_3} &= 3 \end{aligned}$$

By eye we see just what the Theorem says we will see: there is one, and only one, solution $c_1 = 3$, $c_2 = -1$, and $c_3 = -1$.

1.13 *Definition*   In a vector space with basis B the *representation of $\vec{v}$ with respect to* B is the column vector of the coefficients used to express $\vec{v}$ as a linear combination of the basis vectors:

$$\text{Rep}_B(\vec{v}) = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

where $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ and $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$. The c's are the *coordinates of $\vec{v}$ with respect to* B.

Example Above we saw that in $\mathcal{P}_1 = \{a + bx \mid a, b \in \mathbb{R}\}$ one basis is $B = \langle 1 + x, 1 - x \rangle$. As part of that we computed the coefficients needed to express a member of $\mathcal{P}_1$ as a combination of basis vectors.

$$a + bx = c_1(1 + x) + c_2(1 - x) \implies c_1 = (a + b)/2, \ c_2 = (a - b)/2$$

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$$a + bx = c_1(1 + x) + c_2(1 - x) \implies c_1 = (a + b)/2, \ c_2 = (a - b)/2$$

For instance, the polynomial $3 + 4x$ has this expression

$$3 + 4x = (7/2) \cdot (1 + x) + (-1/2) \cdot (1 - x)$$

so its representation is this.

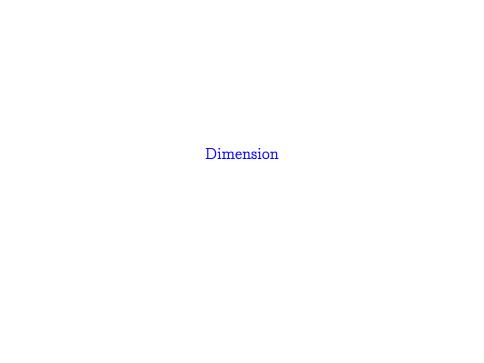$$\text{Rep}_B(3 + 4x) = \begin{pmatrix} 7/2 \\ -1/2 \end{pmatrix}$$

Example With respect to $\mathbb{R}^3$'s standard basis $\mathcal{E}_3$ the vector

$$\vec{v} = \begin{pmatrix} 2 \\ -3 \\ 1/2 \end{pmatrix}$$

has this representation.

$$\mathrm{Rep}_{\mathcal{E}_3}(\vec{v}) = \begin{pmatrix} 2 \\ -3 \\ 1/2 \end{pmatrix}$$

In general, any $\vec{w} \in \mathbb{R}^n$ has $\mathrm{Rep}_{\mathcal{E}_n}(\vec{w}) = \vec{w}$.

Dimension

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2.1 *Definition*   A vector space is *finite-dimensional* if it has a basis with only finitely many vectors.

Example The space $\mathbb{R}^3$ is finite-dimensional since it has a basis with three elements $\mathcal{E}_3$.

Example The space of quadratic polynomials $\mathcal{P}_2$ has at least one basis with finitely many elements, $\langle 1, x, x^2 \rangle$, so it is finite-dimensional.

Example The space $\mathcal{M}_{2\times 2}$ of 2×2 matrices is finite-dimensional. Here is one basis with finitely many members.

$$\langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rangle$$

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Note From this point on we will restrict our attention to vector spaces that are finite-dimensional. All the later examples, definitions, and theorems assume this of the spaces.

We will show that for any finite-dimensional space, all of its bases have the same number of elements.

Example Each of these is a basis for $\mathcal{P}_2$.

$$B_0 = \langle 1, 1 + x, 1 + x + x^2 \rangle$$
$$B_1 = \langle 1 + x + x^2, 1 + x, 1 \rangle$$
$$B_2 = \langle x^2, 1 + x, 1 - x \rangle$$
$$B_3 = \langle 1, x, x^2 \rangle$$

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Example Here are two different bases for $\mathcal{M}_{2\times2}$.

$$B_0 = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \rangle$$
$$B_1 = \langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \rangle$$

Both have four elements.

Exchange Lemma

Example This is a basis for $\mathbb{R}^3$.

$$B = \langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$$

Here are the representations of two vectors with respect to B.

$$\vec{v}_1 = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} = (5/2) \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (1/2) \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\vec{v}_2 = \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} = (3/2) \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + (3/2) \cdot \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} + 2 \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

For the first line, the coefficient of the third vector is 0 so $\vec{v}_1$ has no part in the direction of the third vector. In the second line, the coefficient of the third vector is nonzero so $\vec{v}_2$ has some part in the direction of the third vector of B.

Consider what results when we exchange this third vector from B for $\vec{v}_1$ and $\vec{v}_2$.

$$B_1 = \langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \rangle \qquad B_2 = \langle \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 0 \\ 2 \end{pmatrix} \rangle$$

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Because $\vec{v}_1$ does not involve the third element of B — that is, because in the reprresentation the coefficient of the thrid vector is 0 — the set B_1 is linearly dependent. But the second set B_2 is linearly independent because in the representation of $\vec{v}_2$ the coefficient of the third vector of B is not zero, so $\vec{v}_2$ is not redundant on the other two members of B_2.

2.4 *Lemma*   Assume that $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ is a basis for a vector space, and that for the vector $\vec{v}$ the relationship $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$ has $c_i \neq 0$. Then exchanging $\vec{\beta}_i$ for $\vec{v}$ yields another basis for the space.

Lemma   Assume that $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ is a basis for a vector space, and that for the vector $\vec{v}$ the relationship $\vec{v} = c_1\vec{\beta}_1 + c_2\vec{\beta}_2 + \cdots + c_n\vec{\beta}_n$ has $c_i \neq 0$. Then exchanging $\vec{\beta}_i$ for $\vec{v}$ yields another basis for the space.

Proof   Call the outcome of the exchange $\hat{B} = \langle \vec{\beta}_1, \ldots, \vec{\beta}_{i-1}, \vec{v}, \vec{\beta}_{i+1}, \ldots, \vec{\beta}_n \rangle$.

We first show that $\hat{B}$ is linearly independent. Any relationship $d_1\vec{\beta}_1 + \cdots + d_i\vec{v} + \cdots + d_n\vec{\beta}_n = \vec{0}$ among the members of $\hat{B}$, after substitution for $\vec{v}$,

$$d_1\vec{\beta}_1 + \cdots + d_i \cdot (c_1\vec{\beta}_1 + \cdots + c_i\vec{\beta}_i + \cdots + c_n\vec{\beta}_n) + \cdots + d_n\vec{\beta}_n = \vec{0} \qquad (*)$$

gives a linear relationship among the members of B. The basis B is linearly independent so the coefficient $d_i c_i$ of $\vec{\beta}_i$ is zero. Because we assumed that c_i is nonzero, $d_i = 0$. Using this in equation $(*)$ gives that all of the other d's are also zero. Therefore $\hat{B}$ is linearly independent.

We finish by showing that $\hat{B}$ has the same span as B. Half of this argument, that $[\hat{B}] \subseteq [B]$, is easy; we can write any member $d_1\vec{\beta}_1 + \cdots + d_i\vec{v} + \cdots + d_n\vec{\beta}_n$ of $[\hat{B}]$ as $d_1\vec{\beta}_1 + \cdots + d_i \cdot (c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n) + \cdots + d_n\vec{\beta}_n$, which is a linear combination of linear combinations of members of B, and hence is in $[B]$. For the $[B] \subseteq [\hat{B}]$ half of the argument, recall that if $\vec{v} = c_1\vec{\beta}_1 + \cdots + c_n\vec{\beta}_n$ with $c_i \neq 0$ then we can rearrange the equation to $\vec{\beta}_i = (-c_1/c_i)\vec{\beta}_1 + \cdots + (1/c_i)\vec{v} + \cdots + (-c_n/c_i)\vec{\beta}_n$. Now, consider any member $d_1\vec{\beta}_1 + \cdots + d_i\vec{\beta}_i + \cdots + d_n\vec{\beta}_n$ of $[B]$, substitute for $\vec{\beta}_i$ its expression as a linear combination of the members of $\hat{B}$, and recognize, as in the first half of this argument, that the result is a linear combination of linear combinations of members of $\hat{B}$, and hence is in $[\hat{B}]$.   QED

All of a space's bases are the same size

2.5 *Theorem*   In any finite-dimensional vector space, all bases have the same number of elements.

Example The idea of the proof is that, given two bases, exchange members of the second for members of the first, which shows that they have the same number of elements. For an illustration, these are bases for $\mathcal{P}_2$.

$$B = \langle 1 + x + x^2, 1 + x, 1 \rangle \qquad D = \langle 2, 2x, 2x^2 \rangle$$

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For the first element of D, represent it with respect to B, look for a nonzero coefficient, and exchange.

$$2 = 0 \cdot (1 + x + x^2) + 0 \cdot (1 + x) + 2 \cdot (1) \quad B_1 = \langle 1 + x + x^2, 1 + x, 2 \rangle$$

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Iterate (only exchanging for elements of B).

$$2x = 0 \cdot (1 + x + x^2) + 2 \cdot (1 + x) - 1 \cdot (2) \quad B_2 = \langle 1 + x + x^2, 2x, 2 \rangle$$

The third exchange finishes it off.

$$2x^2 = 2 \cdot (1 + x + x^2) - 1 \cdot (2x) - 1 \cdot (2) \quad B_3 = D$$

Proof   Fix a vector space with at least one finite basis. Choose, from among all of this space's bases, one $B = \langle \vec{\beta}_1, \ldots, \vec{\beta}_n \rangle$ of minimal size. We will show that any other basis $D = \langle \vec{\delta}_1, \vec{\delta}_2, \ldots \rangle$ also has the same number of members, n. Because B has minimal size, D has no fewer than n vectors. We will argue that it cannot have more than n vectors.

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The basis B spans the space and $\vec{\delta}_1$ is in the space, so $\vec{\delta}_1$ is a nontrivial linear combination of elements of B. By the Exchange Lemma, we can swap $\vec{\delta}_1$ for a vector from B, resulting in a basis B_1, where one element is $\vec{\delta}_1$ and all of the $n-1$ other elements are $\vec{\beta}$'s.

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The prior paragraph forms the basis step for an induction argument. The inductive step starts with a basis B_k (for $1 \leqslant k < n$) containing k members of D and $n - k$ members of B. We know that D has at least n members so there is a $\vec{\delta}_{k+1}$. Represent it as a linear combination of elements of B_k. The key point: in that representation, at least one of the nonzero scalars must be associated with a $\vec{\beta}_i$ or else that representation would be a nontrivial linear relationship among elements of the linearly independent set D. Exchange $\vec{\delta}_{k+1}$ for $\vec{\beta}_i$ to get a new basis B_{k+1} with one $\vec{\delta}$ more and one $\vec{\beta}$ fewer than the previous basis B_k.

Repeat that until no $\vec{\beta}$'s remain, so that B_n contains $\vec{\delta}_1, \ldots, \vec{\delta}_n$. Now, D cannot have more than these n vectors because any $\vec{\delta}_{n+1}$ that remains would be in the span of B_n (since it is a basis) and hence would be a linear combination of the other $\vec{\delta}$'s, contradicting that D is linearly independent.

QED

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Example The vector space $\mathcal{M}_{n \times m}$ has dimension $n \cdot m$. A natural basis consists of matrices with a single 1 and the other entries 0's.

Example The solution set S of this system

$$\begin{aligned} x - y + z &= 0 \\ -x + 2y - z + 2w &= 0 \\ -x + 3y - z + 4w &= 0 \end{aligned}$$

is a vector space (this is easy to check for any homogeneous system).
Solving the system

$$\begin{pmatrix} 1 & -1 & 1 & 0 & | & 0 \\ -1 & 2 & -1 & 2 & | & 0 \\ 1 & 3 & -1 & 4 & | & 0 \end{pmatrix} \xrightarrow[\rho_1 + \rho_3]{\rho_1 + \rho_2} \xrightarrow{-2\rho_2 + \rho_3} \begin{pmatrix} 1 & -1 & 1 & 0 & | & 0 \\ 0 & 1 & 0 & 2 & | & 0 \\ 0 & 0 & 0 & 0 & | & 0 \end{pmatrix}$$

and parametrizing gives a basis of two vectors.

$$\{\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \cdot z + \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix} \cdot w \mid z, w \in \mathbb{R}\} \qquad B = \langle \begin{pmatrix} -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -2 \\ -2 \\ 0 \\ 1 \end{pmatrix} \rangle$$

So S is a vector space of dimension two.

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Remark This is an example of a result that assumes the vector spaces are finite-dimensional without specifically saying so.

Corollary   Any linearly independent set can be expanded to make a basis.

Proof   If a linearly independent set is not already a basis then it must not span the space. Adding to the set a vector that is not in the span will preserve linear independence by Lemma II.1.14 . Keep adding until the resulting set does span the space, which the prior corollary shows will happen after only a finite number of steps.   QED

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2.14 *Corollary*   Any spanning set can be shrunk to a basis.

Proof   Call the spanning set S. If S is empty then it is already a basis (the space must be a trivial space). If $S = \{\vec{0}\}$ then it can be shrunk to the empty basis, thereby making it linearly independent, without changing its span.

Otherwise, S contains a vector $\vec{s}_1$ with $\vec{s}_1 \neq \vec{0}$ and we can form a basis $B_1 = \langle \vec{s}_1 \rangle$. If $[B_1] = [S]$ then we are done. If not then there is a $\vec{s}_2 \in [S]$ such that $\vec{s}_2 \notin [B_1]$. Let $B_2 = \langle \vec{s}_1, \vec{s}_2 \rangle$; by Lemma II.1.14 this is linearly independent so if $[B_2] = [S]$ then we are done.

We can repeat this process until the spans are equal, which must happen in at most finitely many steps.   QED

2.15 *Corollary*   In an n-dimensional space, a set composed of n vectors is linearly independent if and only if it spans the space.

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To finish, we will show that any subset with n vectors spans the space if and only if it is a basis. Again, 'if' is trivial. 'Only if' holds because any spanning set can be shrunk to a basis, but a basis has n elements and so this shrunken set is just the one we started with.   QED

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Example This clearly spans the space.

$$\langle \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rangle \subseteq \mathbb{R}^3$$

Because it has same number of elements as the dimension of the space, it is therefore a basis.

Vector Spaces and Linear Systems

Row space

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3.3 *Lemma*   If two matrices A and B are related by a row operation

$$A \xrightarrow{\rho_i \leftrightarrow \rho_j} B \quad \text{or} \quad A \xrightarrow{k\rho_i} B \quad \text{or} \quad A \xrightarrow{k\rho_i + \rho_j} B$$

(for $i \neq j$ and $k \neq 0$) then their row spaces are equal. Hence, row-equivalent matrices have the same row space and therefore the same row rank.

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Proof   Corollary One.III.2.4 shows that when $A \longrightarrow B$ then each row of B is a linear combination of the rows of A. That is, in the above terminology, each row of B is an element of the row space of A. Then Rowspace(B) $\subseteq$ Rowspace(A) follows because a member of the set Rowspace(B) is a linear combination of the rows of B, so it is a combination of combinations of the rows of A, and by the Linear Combination Lemma is also a member of Rowspace(A).

For the other set containment, recall Lemma One.III.1.5 , that row operations are reversible so $A \longrightarrow B$ if and only if $B \longrightarrow A$. Then Rowspace$(A) \subseteq$ Rowspace(B) follows as in the previous paragraph.   QED

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3.3 *Lemma*　　The nonzero rows of an echelon form matrix make up a linearly independent set.

Proof　　Lemma One.III.2.5 says that no nonzero row of an echelon form matrix is a linear combination of the other rows. This result restates that using this chapter's terminology.　　　　　　　　　　　　　QED

Example The matrix before Gauss's Method and the matrix after have equal row spaces.

$$M = \begin{pmatrix} 1 & 2 & 1 & 0 & 3 \\ -1 & -2 & 2 & 2 & 0 \\ 2 & 4 & 5 & 2 & 9 \end{pmatrix} \xrightarrow[-2\rho_1+\rho_3]{\rho_1+\rho_2} \xrightarrow{-\rho_2+\rho_3} \begin{pmatrix} 1 & 2 & 1 & 0 & 3 \\ 0 & 0 & 3 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The nonzero rows of the latter matrix form a basis for $\mathrm{Rowspace}(M)$.

$$B = \langle (1\ \ 2\ \ 1\ \ 0\ \ 3),\ (0\ \ 0\ \ 3\ \ 2\ \ 3)\rangle$$

The row rank is 2.

So Gauss's Method produces a basis for the row space of a matrix. It has found the "repeat" information, that M's third row is three times the first plus the second, and eliminated that extra row.

Column space

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Example This system

$$2x + \quad 3y = d_1$$
$$-x + (1/2)y = d_2$$

has a solution for those $d_1, d_2 \in \mathbb{R}$ that we can find to satisfy this vector equation.

$$x \cdot \begin{pmatrix} 2 \\ -1 \end{pmatrix} + y \cdot \begin{pmatrix} 3 \\ 1/2 \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} \qquad x, y \in \mathbb{R}$$

That is, the system has a solution if and only if the vector on the right is in the column space of this matrix.

$$\begin{pmatrix} 2 & 3 \\ -1 & 1/2 \end{pmatrix}$$

Transpose

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Example To find a basis for the column space of a matrix,

$$\begin{pmatrix} 2 & 3 \\ -1 & 1/2 \end{pmatrix}$$

transpose,

$$\begin{pmatrix} 2 & 3 \\ -1 & 1/2 \end{pmatrix}^\mathsf{T} = \begin{pmatrix} 2 & -1 \\ 3 & 1/2 \end{pmatrix}$$

reduce,

$$\begin{pmatrix} 2 & -1 \\ 3 & 1/2 \end{pmatrix} \xrightarrow{(-3/2)\rho_1 + \rho_2} \begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}$$

and transpose back.

$$\begin{pmatrix} 2 & -1 \\ 0 & 2 \end{pmatrix}^\mathsf{T} = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}$$

This basis

$$B = \langle \begin{pmatrix} 2 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \rangle$$

shows that the column space is the entire vector space $\mathbb{R}^2$.

3.10 *Lemma*   Row operations do not change the column rank.

Lemma   Row operations do not change the column rank.

Proof   Restated, if A reduces to B then the column rank of B equals the column rank of A.

This proof will be finished if we show that row operations do not affect linear relationships among columns, because the column rank is the size of the largest set of unrelated columns. That is, we will show that a relationship exists among columns (such as that the fifth column is twice the second plus the fourth) if and only if that relationship exists after the row operation. But this is exactly the first theorem of this book, Theorem One.I.1.5 : in a relationship among columns,

$$c_1 \cdot \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + \cdots + c_n \cdot \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

row operations leave unchanged the set of solutions $(c_1, \ldots, c_n)$.   QED

3.11 *Theorem* For any matrix, the row rank and column rank are equal.

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Proof   Bring the matrix to reduced echelon form. Then the row rank equals the number of leading entries since that equals the number of nonzero rows. Then also, the number of leading entries equals the column rank because the set of columns containing leading entries consists of some of the $\vec{e}_i$'s from a standard basis, and that set is linearly independent and spans the set of columns. Hence, in the reduced echelon form matrix, the row rank equals the column rank, because each equals the number of leading entries.

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But Lemma 3.3 and Lemma 3.10 show that the row rank and column rank are not changed by using row operations to get to reduced echelon form. Thus the row rank and the column rank of the original matrix are also equal.   QED

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But Lemma 3.3 and Lemma 3.10 show that the row rank and column rank are not changed by using row operations to get to reduced echelon form. Thus the row rank and the column rank of the original matrix are also equal.   QED

3.12 *Definition*   The *rank* of a matrix is its row rank or column rank.

Example The column rank of this matrix

$$\begin{pmatrix} 2 & -1 & 3 & 1 & 0 & 1 \\ 3 & 0 & 1 & 1 & 4 & -1 \\ 4 & -2 & 6 & 2 & 0 & 2 \\ 1 & 0 & 3 & 0 & 0 & 2 \end{pmatrix}$$

is 3. Its largest set of linearly independent columns is size 3 because that's the size of its largest set of linearly independent rows.

$$\xrightarrow[\substack{-2\rho_1+\rho_3 \\ -(1/2)\rho_1+\rho_4}]{-(3/2)\rho_1+\rho_2} \quad \xrightarrow{-(1/3)\rho_2+\rho_4} \quad \xrightarrow{\rho_3 \leftrightarrow \rho_4} \quad \begin{pmatrix} 2 & -1 & 3 & 1 & 0 & 1 \\ 0 & 3/2 & -7/2 & -1/2 & 4 & -5/2 \\ 0 & 0 & 8/3 & -1/3 & -4/3 & 7/3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

3.13 *Theorem*   For linear systems with n unknowns and with matrix of coefficients A, the statements

(1) the rank of A is r
(2) the vector space of solutions of the associated homogeneous system has dimension $n - r$

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Proof   The rank of A is r if and only if Gaussian reduction on A ends with r nonzero rows. That's true if and only if echelon form matrices row equivalent to A have r-many leading variables. That in turn holds if and only if there are $n - r$ free variables.   QED

3.14 *Corollary*   Where the matrix A is $n \times n$, these statements

(1) the rank of A is n

(2) A is nonsingular

(3) the rows of A form a linearly independent set

(4) the columns of A form a linearly independent set

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   (1) the rank of A is n
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   (4) the columns of A form a linearly independent set
   (5) any linear system whose matrix of coefficients is A has one and only
       one solution

are equivalent.

Proof   Clearly $(1) \iff (2) \iff (3) \iff (4)$. The last, $(4) \iff (5)$, holds because a set of n column vectors is linearly independent if and only if it is a basis for $\mathbb{R}^n$, but the system

$$c_1 \begin{pmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{pmatrix} + \cdots + c_n \begin{pmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{pmatrix} = \begin{pmatrix} d_1 \\ d_2 \\ \vdots \\ d_m \end{pmatrix}$$

has a unique solution for all choices of $d_1, \ldots, d_n \in \mathbb{R}$ if and only if the vectors of a's on the left form a basis.   QED