Two.II Linear Independence

Linear Algebra

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Definition and examples

Linear independence

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Observe that, although this way of writing one vector as a combination of the others

$$\vec{s}_0 = c_1\vec{s}_1 + c_2\vec{s}_2 + \cdots + c_n\vec{s}_n$$

visually sets off $\vec{s}_0$, algebraically there is nothing special about that vector in that equation. For any $\vec{s}_i$ with a coefficient c_i that is non-0 we can rewrite to isolate $\vec{s}_i$.

$$\vec{s}_i = (1/c_i)\vec{s}_0 + \cdots + (-c_{i-1}/c_i)\vec{s}_{i-1} + (-c_{i+1}/c_i)\vec{s}_{i+1} + \cdots + (-c_n/c_i)\vec{s}_n$$

When we don't want to single out any vector we will instead say that $\vec{s}_0, \vec{s}_1, \ldots, \vec{s}_n$ are in a *linear relationship* and put all of the vectors on the same side.

1.5 *Lemma* A subset S of a vector space is linearly independent if and only if among its elements the only linear relationship $c_1\vec{s}_1 + \cdots + c_n\vec{s}_n = \vec{0}$ (with $\vec{s}_i \neq \vec{s}_j$ for all $i \neq j$) is the trivial one $c_1 = 0, \ldots, c_n = 0$.

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Proof If S is linearly independent then no vector $\vec{s}_i$ is a linear combination of other vectors from S so there is no linear relationship where some of the $\vec{s}$'s have nonzero coefficients.

If S is not linearly independent then some $\vec{s}_i$ is a linear combination $\vec{s}_i = c_1\vec{s}_1 + \cdots + c_{i-1}\vec{s}_{i-1} + c_{i+1}\vec{s}_{i+1} + \cdots + c_n\vec{s}_n$ of other vectors from S. Subtracting $\vec{s}_i$ from both sides gives a relationship involving a nonzero coefficient, the -1 in front of $\vec{s}_i$. QED

So to decide if a list of vectors $\vec{s}_0, \ldots, \vec{s}_n$ is linearly independent, set up the equation $\vec{0} = c_0\vec{s}_0 + \cdots + c_n\vec{s}_n$, and calculate whether it has any solutions, other than the trivial one where all coefficents are zero.

Example This set of vectors in the plane $\mathbb{R}^2$ is linearly independent.

$$\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \}$$

The only solution to this equation

$$c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

is trivial $c_1 = 0$, $c_2 = 0$.

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Example In the vector space of cubic polynomials $\mathcal{P}_3 = \{a_0 + a_1 x + a_2 x^2 + a_3 x^3 \mid a_i \in \mathbb{R}\}$ the set $\{1 - x, 1 + x\}$ is linearly independent. Setting up the equation $c_0(1 - x) + c_1(1 + x) = 0$ and considering the constant term and linear term, leads to this system

$$\begin{aligned} c_0 + c_1 &= 0 \\ -c_0 + c_1 &= 0 \end{aligned}$$

which has only the trivial solution.

Example The nonzero rows of this matrix form a linearly independent set.

$$\begin{pmatrix} 2 & 0 & 1 & -1 \\ 0 & 1 & -3 & 1/2 \\ 0 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

We showed in Lemma One.III.2.5 that in any echelon form matrix the nonzero rows form a linearly independent set.

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Example This subset of $\mathbb{R}^3$ is linearly dependent.

$$\{ \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \\ 6 \end{pmatrix} \}$$

One way to see that is to spot that the third vector is twice the first plus the second. Another way is to solve the linear system

$$\begin{aligned} c_1 - c_2 + c_3 &= 0 \\ c_1 + c_2 + 3c_3 &= 0 \\ 3c_1 + 6c_3 &= 0 \end{aligned}$$

and note that it has more than just the solution $c_1 = c_2 = c_3 = 0$.

1.2 *Lemma* Where V is a vector space, S is a subset of that space, and $\vec{v}$ is an element of that space, $[S \cup \{\vec{v}\}] = [S]$ if and only if $\vec{v} \in [S]$.

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Example The book has the proof; here is an illustration. The span of this set is the xy-plane.

$$P = \{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} \} \subset \mathbb{R}^3$$

If we expand the set by adding a vector $\{\vec{p}_1, \vec{p}_2, \vec{q}\}$ then there are two possibilities.

$$P_0 = \{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \} \qquad P_1 = \{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \}$$

If the new vector is already in the starting span $\vec{q} \in [P]$ then the span is unchanged $[P_0] = [P]$. But if the new vector is outside the starting span $\vec{q} \notin [P]$ then the span grows $[P_1] \supsetneq [P]$.

1.3 *Corollary* For $\vec{v} \in S$, omitting that vector does not shrink the span $[S] = [S - \{\vec{v}\}]$ if and only if it is dependent on other vectors in the set $\vec{v} \in [S]$.

Example These two subsets of $\mathbb{R}^3$ have the same span

$$\{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}, \begin{pmatrix}7\\8\\9\end{pmatrix}\} \qquad \{\begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}4\\5\\6\end{pmatrix}\}$$

because in the first set $\vec{v}_3 = 2\vec{v}_2 - \vec{v}_1$.

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because in the first set $\vec{v}_3 = 2\vec{v}_2 - \vec{v}_1$.

1.13 *Corollary* A set S is linearly independent if and only if for any $\vec{v} \in S$, its removal shrinks the span $[S - \{v\}] \subsetneq [S]$.

Example This is a linearly independent subset of $\mathcal{P}_3$

$$S = \{1 + x, 1 - x, x^2\}$$

Removal of any element, such as if we remove $1 - x$ to get $\hat{S} = \{1 + x, x^2\}$, will make the span smaller: $[\hat{S}] \subsetneq [S]$.

1.14 *Lemma* Suppose that S is linearly independent and that $\vec{v} \notin S$. Then the set $S \cup \{\vec{v}\}$ is linearly independent if and only if $\vec{v} \notin [S]$.

Example The book has the proof; here is an illustration. Consider this linearly independent subset of $\mathcal{P}_2$.

$$S = \{1 - x, 1 + x\}$$

Its span $[S]$ is the set of linear polynomials $\{a + bx \mid a, b \in \mathbb{R}\}$. (To check: consider $a + bx = r_1(1 - x) + r_2(1 + x)$, which gives a linear system with equations $r_1 + r_2 = a$ and $-r_1 + r_2 = b$, having the solution $r_2 = (1/2)a + (1/2)b$ and $r_1 = (1/2)a - (1/2)b$.)

Here are two supersets.

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On the left, adding a linear polynomial just adds "repeat information" so $[S_1] = [S]$ and S_1 is linearly dependent.

The right, with "new information," enlarges the span $[S_2] = \mathcal{P}_2 \supsetneq [S]$ and the new set S_2 is also linearly independent. (To check this, use $a + bx + cx^2 = r_1(1 - x) + r_2(1 + x) + r_3(2 + x^2)$ to get a linear system with solution $r_3 = c$, $r_2 = (1/2)a + (1/2)b$ and $r_1 = (1/2)a - (1/2)b - c$.)

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Proof If $S = \{\vec{s}_1, \ldots, \vec{s}_n\}$ is linearly independent then S itself satisfies the statement, so assume that it is linearly dependent.

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Proof If $S = \{\vec{s}_1, \ldots, \vec{s}_n\}$ is linearly independent then S itself satisfies the statement, so assume that it is linearly dependent.

By the definition of dependent, S contains a vector $\vec{v}_1$ that is a linear combination of the others. Define the set $S_1 = S - \{\vec{v}_1\}$. By Corollary 1.3 the span does not shrink $[S_1] = [S]$.

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By the definition of dependent, S contains a vector $\vec{v}_1$ that is a linear combination of the others. Define the set $S_1 = S - \{\vec{v}_1\}$. By Corollary 1.3 the span does not shrink $[S_1] = [S]$.

If S_1 is linearly independent then we are done. Otherwise iterate: take a vector $\vec{v}_2$ that is a linear combination of other members of S_1 and discard it to derive $S_2 = S_1 - \{\vec{v}_2\}$ such that $[S_2] = [S_1]$. Repeat this until a linearly independent set S_j appears; one must appear eventually because S is finite and the empty set is linearly independent. QED

Example Consider this subset of $\mathbb{R}^2$.

$$S = \{\vec{s}_1, \vec{s}_2, \vec{s}_3, \vec{s}_4, \vec{s}_5\} = \{\begin{pmatrix}2\\2\end{pmatrix}, \begin{pmatrix}3\\3\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix}, \begin{pmatrix}0\\-1\end{pmatrix}, \begin{pmatrix}1\\-1\end{pmatrix}\}$$

The linear relationship

$$r_1\begin{pmatrix}2\\2\end{pmatrix} + r_2\begin{pmatrix}3\\3\end{pmatrix} + r_3\begin{pmatrix}1\\4\end{pmatrix} + r_4\begin{pmatrix}0\\-1\end{pmatrix} + r_5\begin{pmatrix}1\\-1\end{pmatrix} = \begin{pmatrix}0\\0\end{pmatrix}$$

gives a system of equations.

$$\begin{array}{rcl} 2r_1 + 3r_2 + r_3 \phantom{{}-r_4} + r_5 &=& 0 \\ 2r_1 + 3r_2 + 4r_3 - r_4 - r_5 &=& 0 \end{array}$$

$$\xrightarrow{-\rho_1+\rho_2} \begin{array}{rcl} 2r_1 + 3r_2 + r_3 \phantom{{}-r_4} + r_5 &=& 0 \\ +3r_3 - r_4 - 2r_5 &=& 0 \end{array}$$

Parametrize by expressing the leading variables r_1 and r_3 in terms of the free variables.

$$\{\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} -1/6 \\ 0 \\ 1/3 \\ 1 \\ 0 \end{pmatrix} r_4 + \begin{pmatrix} -5/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \end{pmatrix} r_5 \mid r_2, r_4, r_5 \in \mathbb{R}\}$$

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Set $r_5 = 1$ and $r_2 = r_4 = 0$ to get $r_1 = -5/6$ and $r_3 = 2/3$,

$$-\frac{5}{6} \cdot \begin{pmatrix} 2 \\ 2 \end{pmatrix} + 0 \cdot \begin{pmatrix} 3 \\ 3 \end{pmatrix} + \frac{2}{3} \cdot \begin{pmatrix} 1 \\ 4 \end{pmatrix} + 0 \cdot \begin{pmatrix} 0 \\ -1 \end{pmatrix} + 1 \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

showing that $\vec{s}_5$ is in the span of the set $\{\vec{s}_1, \vec{s}_3\}$.

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showing that $\vec{s}_5$ is in the span of the set $\{\vec{s}_1, \vec{s}_3\}$. Similarly, setting $r_4 = 1$ and the other parameters to 0 shows $\vec{s}_4$ is in the span of the set $\{\vec{s}_1, \vec{s}_3\}$. Also, setting $r_2 = 1$ and the other parameters to 0 shows $\vec{s}_2$ is in the span of the same set.

Parametrize by expressing the leading variables r_1 and r_3 in terms of the free variables.

$$\{\begin{pmatrix} r_1 \\ r_2 \\ r_3 \\ r_4 \\ r_5 \end{pmatrix} = \begin{pmatrix} -3/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} r_2 + \begin{pmatrix} -1/6 \\ 0 \\ 1/3 \\ 1 \\ 0 \end{pmatrix} r_4 + \begin{pmatrix} -5/6 \\ 0 \\ 2/3 \\ 0 \\ 1 \end{pmatrix} r_5 \mid r_2, r_4, r_5 \in \mathbb{R}\}$$

Set $r_5 = 1$ and $r_2 = r_4 = 0$ to get $r_1 = -5/6$ and $r_3 = 2/3$,

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showing that $\vec{s}_5$ is in the span of the set $\{\vec{s}_1, \vec{s}_3\}$. Similarly, setting $r_4 = 1$ and the other parameters to 0 shows $\vec{s}_4$ is in the span of the set $\{\vec{s}_1, \vec{s}_3\}$. Also, setting $r_2 = 1$ and the other parameters to 0 shows $\vec{s}_2$ is in the span of the same set. So we can omit the vectors $\vec{s}_2$, $\vec{s}_4$, $\vec{s}_5$ associated with the free variables without shrinking the span.

Parametrize by expressing the leading variables r_1 and r_3 in terms of the free variables.

$$\{\begin{pmatrix}r_1\\r_2\\r_3\\r_4\\r_5\end{pmatrix}=\begin{pmatrix}-3/2\\1\\0\\0\\0\end{pmatrix}r_2+\begin{pmatrix}-1/6\\0\\1/3\\1\\0\end{pmatrix}r_4+\begin{pmatrix}-5/6\\0\\2/3\\0\\1\end{pmatrix}r_5 \mid r_2,r_4,r_5\in\mathbb{R}\}$$

Set $r_5=1$ and $r_2=r_4=0$ to get $r_1=-5/6$ and $r_3=2/3$,

$$-\frac{5}{6}\cdot\begin{pmatrix}2\\2\end{pmatrix}+0\cdot\begin{pmatrix}3\\3\end{pmatrix}+\frac{2}{3}\cdot\begin{pmatrix}1\\4\end{pmatrix}+0\cdot\begin{pmatrix}0\\-1\end{pmatrix}+1\cdot\begin{pmatrix}1\\-1\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}$$

showing that $\vec{s}_5$ is in the span of the set $\{\vec{s}_1,\vec{s}_3\}$. Similarly, setting $r_4=1$ and the other parameters to 0 shows $\vec{s}_4$ is in the span of the set $\{\vec{s}_1,\vec{s}_3\}$. Also, setting $r_2=1$ and the other parameters to 0 shows $\vec{s}_2$ is in the span of the same set. So we can omit the vectors $\vec{s}_2$, $\vec{s}_4$, $\vec{s}_5$ associated with the free variables without shrinking the span. The set $\{\vec{s}_1,\vec{s}_3\}$ is linearly independent and so we cannot omit any members without shrinking the span.

1.18 *Corollary* A subset $S = \{\vec{s}_1, \ldots, \vec{s}_n\}$ of a vector space is linearly dependent if and only if some $\vec{s}_i$ is a linear combination of the vectors $\vec{s}_1$, $\ldots$, $\vec{s}_{i-1}$ listed before it.

1.18 *Corollary* A subset $S = \{\vec{s}_1, \ldots, \vec{s}_n\}$ of a vector space is linearly dependent if and only if some $\vec{s_i}$ is a linear combination of the vectors $\vec{s}_1$, ..., $\vec{s}_{i-1}$ listed before it.

Proof Consider $S_0 = \{\}$, $S_1 = \{\vec{s_1}\}$, $S_2 = \{\vec{s}_1, \vec{s}_2\}$, etc. Some index $i \geqslant 1$ is the first one with $S_{i-1} \cup \{\vec{s}_i\}$ linearly dependent, and there $\vec{s}_i \in [S_{i-1}]$.

QED

Linear independence and subset

1.19 *Lemma* Any subset of a linearly independent set is also linearly independent. Any superset of a linearly dependent set is also linearly dependent.

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This table summarizes the cases.

	$\hat{S} \subset S$	$\hat{S} \supset S$
S *independent*	$\hat{S}$ must be independent	$\hat{S}$ may be either
S *dependent*	$\hat{S}$ may be either	$\hat{S}$ must be dependent

An example of the lower left is that the set S of all vectors in the space $\mathbb{R}^2$ is linearly dependent but the subset $\hat{S}$ consisting of only the unit vector on the x-axis is independent. By interchanging $\hat{S}$ with S that's also an example of the upper right.