Three.IV Matrix Operations

Linear Algebra

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Sums and Scalar Products

Operations on linear functions

Recall that given a linear map $f\colon V \to W$ then the scalar multiple function

$$\vec{v} \overset{r\cdot f}{\longmapsto} r \cdot (\, f(\vec{v})\,)$$

is also a linear map $rf\colon V \to W$.

And where $f, g\colon V \to W$ are linear then the function that is their sum

$$\vec{v} \overset{f+g}{\longmapsto} f(\vec{v}) + g(\vec{v})$$

is again linear $f + g\colon V \to W$.

We will now see how the matrix representation of $\mathrm{Rep}_{B,D}(f)$ is related to that of $\mathrm{Rep}_{B,D}(rf)$, and how the representations of $\mathrm{Rep}_{B,D}(f)$ and $\mathrm{Rep}_{B,D}(g)$ combine to give the representation of $\mathrm{Rep}_{B,D}(f + g)$.

Example Fix a domain V and codomain W with bases $B = \langle \vec{\beta}_1, \vec{\beta}_2 \rangle$ and $D = \langle \vec{\delta}_1, \vec{\delta}_2 \rangle$. Let $h\colon V \to W$ be linear and consider the map $6f\colon V \to W$ given by $\vec{v} \stackrel{6f}{\longmapsto} 6 \cdot h(\vec{v})$.

Where this is the representation of $h(\vec{v})$

$$\mathrm{Rep}_D(\, h(\vec{v})\,) = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

then $6 \cdot h(\vec{v}) = 6 \cdot (w_1\vec{\delta}_1 + w_2\vec{\delta}_2) = (6w_1) \cdot \vec{\delta}_1 + (6w_2) \cdot \vec{\delta}_2$. So the representation of $6h\,(\vec{v})$

$$\mathrm{Rep}_D(6h\,(\vec{v})) = \begin{pmatrix} 6w_1 \\ 6w_2 \end{pmatrix}$$

is entry-by-entry bigger by a factor of 6.

For instance, if

$$H = \mathrm{Rep}_{B,D}(h) = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}$$

then this represents the action of h.

$$\mathrm{Rep}_D(h(\vec{v})) = \mathrm{Rep}_{B,D}(h)\ \mathrm{Rep}_B(\vec{v}) = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 2v_1 + v_2 \\ 3v_1 + 4v_2 \end{pmatrix}$$

By the prior slide we know that the output of $6h$ is 6 times bigger.

$$\mathrm{Rep}_D(6h\,(\vec{v})) = \mathrm{Rep}_{B,D}(6h)\ \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 12v_1 + 6v_2 \\ 18v_1 + 24v_2 \end{pmatrix}$$

So the matrix representing $6h$

$$\mathrm{Rep}_{B,D}(6h) = \begin{pmatrix} 12 & 6 \\ 18 & 24 \end{pmatrix}$$

is entry-by-entry 6 times as large as the matrix representing h.

Example Next consider the representation of the sum $f+g$ of two linear maps $f,g\colon V\to W$. For a domain vector $\vec{v}$ let the outputs $f(\vec{v})$ and $g(\vec{v})$ have these representations.

$$\operatorname{Rep}_D(f(\vec{v}))=\begin{pmatrix}u_1\\u_2\end{pmatrix}\qquad\operatorname{Rep}_D(g(\vec{v}))=\begin{pmatrix}w_1\\w_2\end{pmatrix}$$

The action of $f+g$ is this.

$$\begin{aligned}\vec{v}\stackrel{f+g}{\longmapsto}\vec{u}+\vec{v}&=(u_1\vec{\delta}_1+u_2\vec{\delta}_2)+(w_1\vec{\delta}_1+w_2\vec{\delta}_2)\\&=(u_1+w_1)\cdot\vec{\delta}_1+(u_2+w_2)\cdot\vec{\delta}_2\end{aligned}$$

The effect on the representations of adding the functions is to add the column vectors.

$$\operatorname{Rep}_D(\,(f+g)\,(\vec{v})\,)=\begin{pmatrix}u_1+w_1\\u_2+w_2\end{pmatrix}$$

For instance, let these be the map representations.

$$\mathrm{Rep}_{B,D}(f)=\begin{pmatrix}2&1\\3&4\end{pmatrix}\qquad \mathrm{Rep}_{B,D}(g)=\begin{pmatrix}5&8\\7&6\end{pmatrix}$$

The functions given have this effect.

$$\mathrm{Rep}_{D}(f(\vec{v}))=\begin{pmatrix}2&1\\3&4\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}2v_1+v_2\\3v_1+4v_2\end{pmatrix}$$

$$\mathrm{Rep}_{D}(g(\vec{v}))=\begin{pmatrix}5&8\\7&6\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}5v_1+8v_2\\7v_1+6v_2\end{pmatrix}$$

The prior slide says that $f+g$ acts in this way

$$\mathrm{Rep}_{D}(\,(f+g)\,(\vec{v})\,)\begin{pmatrix}v_1\\v_2\end{pmatrix}=\begin{pmatrix}7v_1+9v_2\\10v_1+10v_2\end{pmatrix}$$

so its matrix

$$\mathrm{Rep}_{B,D}(f+g)=\begin{pmatrix}7&9\\10&10\end{pmatrix}$$

is the entry-by-entry sum of the other two.

Definition of matrix sum and scalar multiple

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Example Where

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 & 2 \\ 9 & -1/2 & 5 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 8 & -1 \end{pmatrix}$$

Then

$$A + C = \begin{pmatrix} 2 & -1 \\ 10 & 2 \end{pmatrix} \qquad 5B = \begin{pmatrix} 0 & 0 & 10 \\ 45 & -5/2 & 25 \end{pmatrix}$$

Note that none of these is defined: $A + B$, $B + A$, $B + C$, $C + B$.

From the definition, they are not defined because the sizes don't match and so the entry-by-entry sum is not possible. But really they are not defined because the underlying function operations are not possible. The fact that A has two columns means that functions represented by A have two-dimensional domains. Functions represented by B have three-dimensional domains. Adding the two functions would be adding apples and oranges.

1.4 *Theorem* Let $h, g\colon V \to W$ be linear maps represented with respect to bases B, D by the matrices H and G and let r be a scalar. Then with respect to B, D the map $r \cdot h\colon V \to W$ is represented by rH and the map $h + g\colon V \to W$ is represented by $H + G$.

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Example The zero matrix is the identity element for matrix addition.

$$\begin{pmatrix} 3 & 1 & 2 \\ 5 & 0 & 9 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ 5 & 0 & 9 \end{pmatrix}$$

A zero function $Z\colon V \to W$ is the identity element for function addition, and the matrix fact accords with the map fact.

Matrix Multiplication

Representing composition

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Proof Let $h\colon V \to W$ and $g\colon W \to U$ be linear. The calculation

$$\begin{aligned} g \circ h\,\bigl(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2\bigr) &= g\bigl(\,h(c_1 \cdot \vec{v}_1 + c_2 \cdot \vec{v}_2)\,\bigr) = g\bigl(\,c_1 \cdot h(\vec{v}_1) + c_2 \cdot h(\vec{v}_2)\,\bigr) \\ &= c_1 \cdot g\bigl(h(\vec{v}_1)\bigr) + c_2 \cdot g\bigl(h(\vec{v}_2)\bigr) = c_1 \cdot (g \circ h)(\vec{v}_1) + c_2 \cdot (g \circ h)(\vec{v}_2) \end{aligned}$$

shows that $g \circ h\colon V \to U$ preserves linear combinations, and so is linear.

QED

Example Consider two linear functions $h\colon V \to W$ and $g\colon W \to X$ represented as here.

$$\mathrm{Rep}_{B,C}(h) = \begin{pmatrix} 3 & 1 \\ 2 & 5 \\ 4 & 6 \end{pmatrix} \qquad \mathrm{Rep}_{C,D}(g) = \begin{pmatrix} 8 & 7 & 11 \\ 9 & 10 & 12 \end{pmatrix}$$

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We start with the action of h on $\vec{v} \in V$.

$$\begin{aligned} \text{Rep}_C(h(\vec{v})) &= \text{Rep}_{B,C}(h) \cdot \text{Rep}_B(\vec{v}) \\ &= \begin{pmatrix} 3 & 1 \\ 2 & 5 \\ 4 & 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 3v_1 + v_2 \\ 2v_1 + 5v_2 \\ 4v_1 + 6v_2 \end{pmatrix} \end{aligned}$$

Next, to that apply g.

$$\mathrm{Rep}_{C,D}(g) \cdot \mathrm{Rep}_C(h(\vec{v})) = \begin{pmatrix} 8 & 7 & 11 \\ 9 & 10 & 12 \end{pmatrix} \begin{pmatrix} 3v_1 + v_2 \\ 2v_1 + 5v_2 \\ 4v_1 + 6v_2 \end{pmatrix}$$

$$= \begin{pmatrix} 8(3v_1 + v_2) + 7(2v_1 + 5v_2) + 11(4v_1 + 6v_2) \\ 9(3v_1 + v_2) + 10(2v_1 + 5v_2) + 12(4v_1 + 6v_2) \end{pmatrix}$$

Gather terms.

$$= \begin{pmatrix} (8 \cdot 3 + 7 \cdot 2 + 11 \cdot 4)v_1 + (8 \cdot 1 + 7 \cdot 5 + 11 \cdot 6)v_2 \\ (9 \cdot 3 + 10 \cdot 2 + 12 \cdot 4)v_1 + (9 \cdot 1 + 10 \cdot 5 + 12 \cdot 6)v_2 \end{pmatrix}$$

Rewrite as a matrix-vector multiplication.

$$= \begin{pmatrix} 8 \cdot 3 + 7 \cdot 2 + 11 \cdot 4 & 8 \cdot 1 + 7 \cdot 5 + 11 \cdot 6 \\ 9 \cdot 3 + 10 \cdot 2 + 12 \cdot 4 & 9 \cdot 1 + 10 \cdot 5 + 12 \cdot 6 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

So here is how the two starting matrices combine.

$$\begin{pmatrix} 8 & 7 & 11 \\ 9 & 10 & 12 \end{pmatrix} \begin{pmatrix} 3 & 1 \\ 2 & 5 \\ 4 & 6 \end{pmatrix} = \begin{pmatrix} 8 \cdot 3 + 7 \cdot 2 + 11 \cdot 4 & 8 \cdot 1 + 7 \cdot 5 + 11 \cdot 6 \\ 9 \cdot 3 + 10 \cdot 2 + 12 \cdot 4 & 9 \cdot 1 + 10 \cdot 5 + 12 \cdot 6 \end{pmatrix}$$

Definition of matrix multiplication

2.3 *Definition* The *matrix-multiplicative product* of the $m \times r$ matrix G and the $r \times n$ matrix H is the $m \times n$ matrix P, where

$$p_{i,j} = g_{i,1}h_{1,j} + g_{i,2}h_{2,j} + \cdots + g_{i,r}h_{r,j}$$

so that the i, j-th entry of the product is the dot product of the i-th row of the first matrix with the j-th column of the second.

$$GH = \begin{pmatrix} & \vdots & & \\ g_{i,1} & g_{i,2} & \cdots & g_{i,r} \\ & \vdots & & \end{pmatrix} \begin{pmatrix} & h_{1,j} & \\ \cdots & h_{2,j} & \cdots \\ & \vdots & \\ & h_{r,j} & \end{pmatrix} = \begin{pmatrix} & \vdots & \\ \cdots & p_{i,j} & \cdots \\ & \vdots & \end{pmatrix}$$

Example

$$\begin{pmatrix} 3 & 1 & 6 \\ 2 & 5 & 9 \end{pmatrix} \begin{pmatrix} 2 & 0 & 4 \\ 1 & -3 & 5 \\ 4 & 2 & 7 \end{pmatrix} = \begin{pmatrix} 31 & 9 & 59 \\ 45 & 3 & 96 \end{pmatrix}$$

Example This product

$$\begin{pmatrix}1 & 3 & -1\\0 & 0 & 0\\2 & 0 & 0\end{pmatrix}\begin{pmatrix}5 & 7 & 1\\2 & 2 & 0\end{pmatrix}$$

is not defined because the number of columns on the left must equal the number of rows on the right.

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Example Square matrices of the same size have a defined product.

$$\begin{pmatrix} 1 & 3 & -1 \\ 0 & 0 & 0 \\ 2 & 0 & 0 \end{pmatrix} \begin{pmatrix} 5 & 7 & 1 \\ 2 & 2 & 0 \\ 1 & -1 & 2 \end{pmatrix} = \begin{pmatrix} 10 & 14 & -1 \\ 0 & 0 & 0 \\ 10 & 14 & 2 \end{pmatrix}$$

This reflects the fact that we can compose two functions from a space to itself $g, h \colon V \to V$.

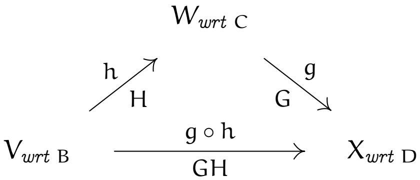
Matrix multiplication represents composition

2.7 *Theorem* A composition of linear maps is represented by the matrix product of the representatives.

The book has the proof, which retraces the steps of the example.

Arrow diagrams

This pictures the relationship between maps and matrices.



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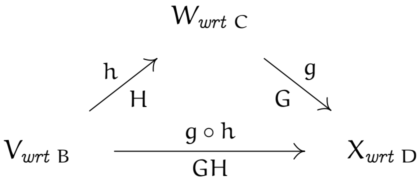

Above the arrows, the maps show that the two ways of going from V to X, straight over via the composition or else in two steps by way of W, have the same effect

$$\vec{v} \stackrel{g\circ h}{\longmapsto} g(h(\vec{v})) \qquad \vec{v} \stackrel{h}{\longmapsto} h(\vec{v}) \stackrel{g}{\longmapsto} g(h(\vec{v}))$$

(this is just the definition of composition). Below the arrows, the matrices indicate that multiplying GH into the column vector $\text{Rep}_B(\vec{v})$ has the same effect as multiplying the column vector first by H and then multiplying the result by G.

$$\text{Rep}_{B,D}(g \circ h) = GH \qquad \text{Rep}_{C,D}(g)\,\text{Rep}_{B,C}(h) = GH$$

Example Let $V = \mathbb{R}^2$, $W = \mathcal{P}_2$, and $X = \mathcal{M}_{2\times 2}$. Fix these bases.

$$B = \langle \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \rangle \qquad C = \langle x^2, x^2 + x, x^2 + x + 1 \rangle$$

$$D = \langle \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} \rangle$$

Suppose that $h\colon \mathbb{R}^2 \to \mathcal{P}_2$ and $g\colon \mathcal{P}_2 \to \mathcal{M}_{2\times 2}$ have these actions.

$$\begin{pmatrix} a \\ b \end{pmatrix} \overset{h}{\longmapsto} ax^2 + (a + b) \qquad px^2 + qx + r \overset{g}{\longmapsto} \begin{pmatrix} p & p + q \\ 0 & r \end{pmatrix}$$

Then the composition does this.

$$\begin{pmatrix} a \\ b \end{pmatrix} \overset{h}{\longmapsto} ax^2 + (a + b) \overset{g}{\longmapsto} \begin{pmatrix} a & a \\ 0 & a + b \end{pmatrix}$$

Here is the same statement in the other notation.

$$g \circ h \left(\begin{pmatrix} a \\ b \end{pmatrix} \right) = \begin{pmatrix} a & a \\ 0 & a + b \end{pmatrix}$$

We next compute the matrices representing those maps, and we will finish by checking that the product of G and H is the matrix representing $g \circ h$.

To find $H = \text{Rep}_{B,D}(h)$, compute the action of h on the domain basis vectors,

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \stackrel{h}{\longmapsto} x^2 + 2 \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix} \stackrel{h}{\longmapsto} x^2$$

represent the results with respect to D, and make the matrix.

$$\text{Rep}_C(x^2 + 2) = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} \quad \text{Rep}_C(x^2) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad H = \begin{pmatrix} 1 & 1 \\ -2 & 0 \\ 2 & 0 \end{pmatrix}$$

Do the same for g: see where it maps its domain basis

$$x^2 \mapsto \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad x^2 + x \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \quad x^2 + x + 1 \mapsto \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

and represent those with respect to its codomain basis.

$$G = \begin{pmatrix} 1 & 1 & 1 \\ 1/2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 1/4 \end{pmatrix}$$

Next, $g \circ h$ has this action.

$$\begin{pmatrix}1\\1\end{pmatrix} \mapsto \begin{pmatrix}1&1\\0&2\end{pmatrix} \qquad \begin{pmatrix}1\\-1\end{pmatrix} \mapsto \begin{pmatrix}1&1\\0&0\end{pmatrix}$$

This represents the composition.

$$\text{Rep}_{B,D}(g \circ h) = \begin{pmatrix}1&1\\1/2&1/2\\0&0\\1/2&0\end{pmatrix}$$

Finish by checking that the product of G with H equals the matrix representing $g \circ h$.

$$\text{Rep}_{C,D}(g) \cdot \text{Rep}_{B,C}(h) = \text{Rep}_{B,D}(g \circ h)$$
$$\begin{pmatrix}1&1&1\\1/2&1&1\\0&0&0\\0&0&1/4\end{pmatrix}\begin{pmatrix}1&1\\-2&0\\2&0\end{pmatrix} = \begin{pmatrix}1&1\\1/2&1/2\\0&0\\1/2&0\end{pmatrix}$$

Order, dimensions, and sizes

An important observation about the order in which we write these things: in writing the composition $g \circ h$, the function g is written first, that is, leftmost, but it is applied second.

$$\vec{v} \stackrel{h}{\longmapsto} h(\vec{v}) \stackrel{g}{\longmapsto} g(h(\vec{v}))$$

That order carries over to matrices: $g \circ h$ is represented by GH.

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That order carries over to matrices: $g \circ h$ is represented by GH.

Also consider the dimensions of the spaces.

$$\text{dimension } n \text{ space} \xrightarrow{h} \text{dimension } r \text{ space} \xrightarrow{g} \text{dimension } m \text{ space}$$

Briefly, $m \times r$ times $r \times n$ equals $m \times n$, as here.

$$\begin{array}{cccc} 2\times 3 & 3\times 4 & = & 2\times 4 \\ \begin{pmatrix} 2 & 1 & 4 \\ -1 & 0 & 3 \end{pmatrix} & \begin{pmatrix} 3 & 0 & 2 & 1 \\ 5 & 0 & 0 & 2 \\ 1 & -1 & 4 & 7 \end{pmatrix} & = & \begin{pmatrix} 15 & -4 & 20 & 32 \\ 0 & -3 & 10 & 20 \end{pmatrix} \end{array}$$

Matrix multiplication is not commutative

Function composition is in general not a commutative operation — $\cos(\sqrt{x})$ is different than $\sqrt{\cos(x)}$. This holds even in the special case of composition of linear functions.

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Example Changing the order in which we multiply these matrices

$$\begin{pmatrix} 3 & 3 \\ 0 & 4 \end{pmatrix} \begin{pmatrix} -2 & 6 \\ 6 & 5 \end{pmatrix} = \begin{pmatrix} 12 & 33 \\ 24 & 20 \end{pmatrix}$$

changes the result.

$$\begin{pmatrix} -2 & 6 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 3 & 3 \\ 0 & 4 \end{pmatrix} = \begin{pmatrix} -6 & 18 \\ 18 & 38 \end{pmatrix}$$

Example The product of these two is defined in one order and not defined in the other.

$$\begin{pmatrix} 3 & 4 \\ 0 & 2 \end{pmatrix} \qquad \begin{pmatrix} 8 & 12 & 0 \\ -4 & 0 & 1/2 \end{pmatrix}$$

Although the matrix operation of multiplication does not have the property of being commutative, it does have some nice algebraic properties.

2.12 *Theorem* If F, G, and H are matrices, and the matrix products are defined, then the product is associative $(FG)H = F(GH)$ and distributes over matrix addition $F(G+H) = FG + FH$ and $(G+H)F = GF + HF$.

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2.12 *Theorem* If F, G, and H are matrices, and the matrix products are defined, then the product is associative $(FG)H = F(GH)$ and distributes over matrix addition $F(G+H) = FG + FH$ and $(G+H)F = GF + HF$.

Proof Associativity holds because matrix multiplication represents function composition, which is associative: the maps $(f \circ g) \circ h$ and $f \circ (g \circ h)$ are equal as both send $\vec{v}$ to $f(g(h(\vec{v})))$.

Distributivity is similar. For instance, the first one goes $f \circ (g+h)\,(\vec{v}) = f\big(\,(g+h)(\vec{v})\,\big) = f\big(\,g(\vec{v}) + h(\vec{v})\,\big) = f(g(\vec{v})) + f(h(\vec{v})) = f \circ g(\vec{v}) + f \circ h(\vec{v})$ (the third equality uses the linearity of f). Right-distributivity goes the same way. QED

Mechanics of Matrix Multiplication

Combinatorics of multiplication

The striking thing about matrix multiplication is the way rows and columns combine. The i, j entry of the matrix product GH is the dot product of row i of the left matrix G with column j of the right one H.

$$p_{i,j} = g_{i,1}h_{1,j} + g_{i,2}h_{2,j} + \cdots + g_{i,r}h_{r,j}$$

Here a second row and a third column combine to make a $2, 3$ entry.

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 4 & 6 \\ 5 & 7 \end{pmatrix} = \begin{pmatrix} 9 & 13 \\ 5 & 7 \\ 4 & 6 \end{pmatrix}$$

We can view this as the left matrix acting by multiplying its rows into the columns of the right matrix. Or we could see it as the right matrix using its columns to act on the left matrix's rows.

3.7 *Lemma* In a product of two matrices G and H, the columns of GH are formed by taking G times the columns of H

$$G \cdot \begin{pmatrix} \vdots & & \vdots \\ \vec{h}_1 & \cdots & \vec{h}_n \\ \vdots & & \vdots \end{pmatrix} = \begin{pmatrix} \vdots & & \vdots \\ G \cdot \vec{h}_1 & \cdots & G \cdot \vec{h}_n \\ \vdots & & \vdots \end{pmatrix}$$

and the rows of GH are formed by taking the rows of G times H

$$\begin{pmatrix} \cdots & \vec{g}_1 & \cdots \\ & \vdots & \\ \cdots & \vec{g}_r & \cdots \end{pmatrix} \cdot H = \begin{pmatrix} \cdots & \vec{g}_1 \cdot H & \cdots \\ & \vdots & \\ \cdots & \vec{g}_r \cdot H & \cdots \end{pmatrix}$$

(ignoring the extra parentheses).

3.2 *Definition* A matrix with all 0's except for a 1 in the i, j entry is an i, j *unit* matrix (or *matrix unit*).

Example The $2, 1$ unit 2×3 matrix multiplies from the left

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 3 \\ 5 & 6 & 4 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 3 \end{pmatrix}$$

to copy row 1 of the multiplicand into row 2 of the result.

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Example From the right the 2, 1 unit 2×3 matrix

$$\begin{pmatrix} 3 & 4 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 5 & 0 & 0 \end{pmatrix}$$

copies column 2 of the first matrix into column 1 of the result.

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to copy row 1 of the multiplicand into row 2 of the result.

Example From the right the $2, 1$ unit 2×3 matrix

$$\begin{pmatrix} 3 & 4 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 \\ 5 & 0 & 0 \end{pmatrix}$$

copies column 2 of the first matrix into column 1 of the result.

Example Rescaling the unit matrix rescales the result.

$$\begin{pmatrix} 3 & 4 \\ 6 & 5 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 3 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 12 & 0 & 0 \\ 15 & 0 & 0 \end{pmatrix}$$

3.8 *Definition* The *main diagonal* (or *principle diagonal* or *diagonal*) of a square matrix goes from the upper left to the lower right.

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3.9 *Definition* An *identity matrix* is square and every entry is 0 except for 1's in the main diagonal.

$$I_{n\times n} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ 0 & 1 & \ldots & 0 \\ & \vdots & & \\ 0 & 0 & \ldots & 1 \end{pmatrix}$$

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Taking the product with an identity matrix returns the multiplicand.

Example Multiplication by an identity from the left

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix}$$

or from the right leaves the matrix unchanged.

$$\begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix}$$

3.12 *Definition* A *diagonal matrix* is square and has 0's off the main diagonal.

$$\begin{pmatrix} a_{1,1} & 0 & \ldots & 0 \\ 0 & a_{2,2} & \ldots & 0 \\ & \vdots & & \\ 0 & 0 & \ldots & a_{n,n} \end{pmatrix}$$

3.12 *Definition* A *diagonal matrix* is square and has 0's off the main diagonal.

$$\begin{pmatrix} a_{1,1} & 0 & \ldots & 0 \\ 0 & a_{2,2} & \ldots & 0 \\ & \vdots & & \\ 0 & 0 & \ldots & a_{n,n} \end{pmatrix}$$

Example Multiplication from the left by a diagonal matrix rescales the rows.

$$\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ -3 & 15 \end{pmatrix}$$

From the right it rescales the columns.

$$\begin{pmatrix} 3 & 2 \\ -1 & 5 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} = \begin{pmatrix} 6 & 6 \\ -2 & 15 \end{pmatrix}$$

3.14 *Definition* A *permutation matrix* is square and is all 0's except for a single 1 in each row and column.

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Example Multiplication by a permutation matrix from the left will swap rows.

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} = \begin{pmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{pmatrix}$$

From the right it swaps columns.

$$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 5 & 4 & 6 \\ 8 & 7 & 9 \end{pmatrix}$$

3.19 *Definition* The *elementary reduction matrices* (or just *elementary matrices*) result from applying a single Gaussian operation to an identity matrix.

1) $I \xrightarrow{k\rho_i} M_i(k)$ for $k \neq 0$

2) $I \xrightarrow{\rho_i \leftrightarrow \rho_j} P_{i,j}$ for $i \neq j$

3) $I \xrightarrow{k\rho_i + \rho_j} C_{i,j}(k)$ for $i \neq j$

Example Multiplying on the left by the 3×3 matrix $M_2(1/2)$ has the effect of the row operation $(1/2)\rho_2$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 5 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 1 & 5/2 \\ 0 & 0 & 0 \end{pmatrix}$$

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Example Left multiplication by $C_{1,3}(-2)$ performs the row operation $-2\rho_1 + \rho_3$.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 5 \\ 2 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 \\ 0 & 2 & 5 \\ 0 & -5 & 1 \end{pmatrix}$$

3.20 *Lemma* Matrix multiplication can do Gaussian reduction.

1) If $H \xrightarrow{k\rho_i} G$ then $M_i(k)H = G$.

2) If $H \xrightarrow{\rho_i \leftrightarrow \rho_j} G$ then $P_{i,j}H = G$.

3) If $H \xrightarrow{k\rho_i + \rho_j} G$ then $C_{i,j}(k)H = G$.

Proof Clear. QED

3.23 *Corollary* For any matrix H there are elementary reduction matrices R_1, ..., R_r such that $R_r \cdot R_{r-1} \cdots R_1 \cdot H$ is in reduced echelon form.

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3.23 *Corollary* For any matrix H there are elementary reduction matrices R_1, ..., R_r such that $R_r \cdot R_{r-1} \cdots R_1 \cdot H$ is in reduced echelon form.

Example You can do this Gauss-Jordan reduction

$$\begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} \xrightarrow{-4\rho_1 + \rho_2} \xrightarrow{-(1/5)\rho_2} \xrightarrow{-2\rho_2 + \rho_1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

as a product; note that the order is right-to-left.

$$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1/5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -4 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C_{2,1}(-2)\, M_2(-1/5)\, C_{1,2}(-4) \cdot H = I$$

Example You can bring this augmented matrix to echelon form with matrix multiplication.

$$\left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 2 & -2 & -1 & 6 \\ 0 & 3 & 1 & 5 \end{array}\right)$$

First perform $-2\rho_1 + \rho_2$ via left multiplication by $C_{1,2}(-2)$.

$$\begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 2 & -2 & -1 & 6 \\ 0 & 3 & 1 & 5 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 0 & -5 & -2 \\ 0 & 3 & 1 & 5 \end{array}\right)$$

Swap rows 2 and 3 with $P_{2,3}$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 0 & -5 & -2 \\ 0 & 3 & 1 & 5 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & -5 & -2 \end{array}\right)$$

Here is the full equation.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 2 & -2 & -1 & 6 \\ 0 & 3 & 1 & 5 \end{array}\right) = \left(\begin{array}{ccc|c} 1 & -1 & 2 & 4 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & -5 & -2 \end{array}\right)$$

Inverses

Function inverses

We finish this section by considering how to represent the inverse of a linear map. Our goal is, where a linear h has an inverse, to find the relationship between the matrices $\mathrm{Rep}_{B,D}(h)$ and $\mathrm{Rep}_{D,B}(h^{-1})$.

Function inverses

We finish this section by considering how to represent the inverse of a linear map. Our goal is, where a linear h has an inverse, to find the relationship between the matrices $\text{Rep}_{B,D}(h)$ and $\text{Rep}_{D,B}(h^{-1})$.

We first recall some things about inverses. Where $\pi\colon \mathbb{R}^3 \to \mathbb{R}^2$ is the projection map and $\iota\colon \mathbb{R}^2 \to \mathbb{R}^3$ is the embedding

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \overset{\pi}{\longmapsto} \begin{pmatrix} x \\ y \end{pmatrix} \qquad \begin{pmatrix} x \\ y \end{pmatrix} \overset{\iota}{\longmapsto} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix}$$

then the composition $\pi \circ \iota$ is the identity map $\pi \circ \iota = \text{id}$ on $\mathbb{R}^2$.

$$\begin{pmatrix} x \\ y \end{pmatrix} \overset{\iota}{\longmapsto} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} \overset{\pi}{\longmapsto} \begin{pmatrix} x \\ y \end{pmatrix}$$

We say that ι is a *right inverse* of π or, what is the same thing, that π is a *left inverse* of ι.

However, composition in the other order $\iota \circ \pi$ doesn't give the identity map — here is a vector that is not sent to itself under $\iota \circ \pi$.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \stackrel{\iota}{\longmapsto} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

However, composition in the other order $\iota \circ \pi$ doesn't give the identity map — here is a vector that is not sent to itself under $\iota \circ \pi$.

$$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \stackrel{\iota}{\longmapsto} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

In fact, π has no left inverse at all. For, if f were to be a left inverse of π then we would have

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \stackrel{\pi}{\longmapsto} \begin{pmatrix} x \\ y \end{pmatrix} \stackrel{f}{\longmapsto} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

for all of the infinitely many z's. But a function f cannot send a single argument $\binom{x}{y}$ to more than one value.

So a function can have a right inverse but no left inverse, or a left inverse but no right inverse. A function can also fail to have an inverse on either side; one example is the zero transformation on $\mathbb{R}^2$.

Some functions have a *two-sided inverse*, another function that is the inverse both from the left and from the right. For instance, the transformation given by $\vec{v} \mapsto 2 \cdot \vec{v}$ has the two-sided inverse $\vec{v} \mapsto (1/2) \cdot \vec{v}$. The appendix shows that a function has a two-sided inverse if and only if it is both one-to-one and onto. The appendix also shows that if a function f has a two-sided inverse then it is unique, so we call it 'the' inverse and write f^{-1}.

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In addition, recall that we have shown in Theorem II.2.20 that if a linear map has a two-sided inverse then that inverse is also linear.

Definition of matrix inverse

4.1 *Definition* A matrix G is a *left inverse matrix* of the matrix H if GH is the identity matrix. It is a *right inverse* if HG is the identity. A matrix H with a two-sided inverse is an *invertible matrix*. That two-sided inverse is denoted H^{-1}.

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Example This matrix

$$H = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

has a two-sided inverse.

$$H^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

To check that, we multiply them in both orders. Here is one; the other is just as easy.

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Example One advantage of knowing a matrix inverse is that it makes solving a linear system easy and quick. To solve

$$\begin{aligned} 2x + 5y &= -3 \\ x + 3y &= \ 10 \end{aligned}$$

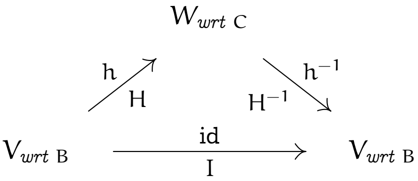
rewrite as a matrix equation

$$\begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -3 \\ 10 \end{pmatrix}$$

and multiply both sides (from the left) by the matrix inverse.

$$\begin{aligned} \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix} \cdot \begin{pmatrix} -3 \\ 10 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -59 \\ 23 \end{pmatrix} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} -59 \\ 23 \end{pmatrix} \end{aligned}$$

This specializes the arrow diagram for composition to the case of inverses.



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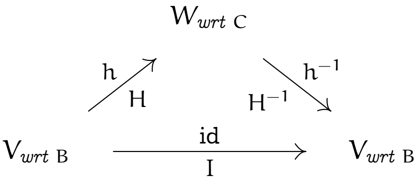

4.2 *Lemma* If a matrix has both a left inverse and a right inverse then the two are equal.

4.3 *Theorem* A matrix is invertible if and only if it is nonsingular.

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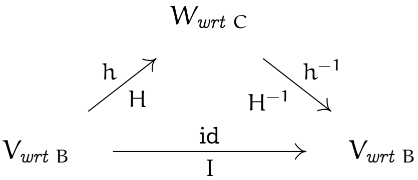

4.2 *Lemma* If a matrix has both a left inverse and a right inverse then the two are equal.

4.3 *Theorem* A matrix is invertible if and only if it is nonsingular.

Proof *(For both results.)* Given a matrix H, fix spaces of appropriate dimension for the domain and codomain and fix bases for these spaces. With respect to these bases, H represents a map h. The statements are true about the map and therefore they are true about the matrix. QED

4.4 *Lemma* A product of invertible matrices is invertible: if G and H are invertible and GH is defined then GH is invertible and $(GH)^{-1} = H^{-1}G^{-1}$.

4.7 *Lemma* A matrix H is invertible if and only if it can be written as the product of elementary reduction matrices. We can compute the inverse by applying to the identity matrix the same row steps, in the same order, that Gauss-Jordan reduce H.

Example This matrix is nonsingular and so is invertible.

$$A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -1 \\ 1 & 2 & 0 \end{pmatrix}$$

To ease the inverse calculation described in the prior proof, we write the matrix A next to the 3×3 identity and as we Gauss-Jordan reduce the matrix on the left, we apply those operations also on the right.

$$\left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & 0 & -1 & 0 & 1 & 0 \\ 1 & 2 & 0 & 0 & 0 & 1 \end{array}\right) \xrightarrow[-\rho_1+\rho_3]{-2\rho_1+\rho_2} \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -6 & -3 & -2 & 1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{-1/6\rho_2+\rho_3} \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -6 & -3 & -2 & 1 & 0 \\ 0 & 0 & -1/2 & -2/3 & -1/6 & 1 \end{array}\right)$$

The right-hand side is in echelon form. We continue with the second half of Gauss-Jordan reduction on the next slide.

$$\xrightarrow[-2\rho_3]{-1/6\rho_2} \left(\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1/2 & 1/3 & -1/6 & 0 \\ 0 & 0 & 1 & 4/3 & 1/3 & -2 \end{array}\right)$$

$$\xrightarrow[-(1/2)\rho_3+\rho_2]{-\rho_3+\rho_1} \left(\begin{array}{ccc|ccc} 1 & 3 & 0 & -1/3 & -1/3 & 2 \\ 0 & 1 & 0 & -1/3 & -1/3 & 1 \\ 0 & 0 & 1 & 4/3 & 1/3 & -2 \end{array}\right)$$

$$\xrightarrow{-3\rho_2+\rho_1} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 2/3 & 2/3 & -1 \\ 0 & 1 & 0 & -1/3 & -1/3 & 1 \\ 0 & 0 & 1 & 4/3 & 1/3 & -2 \end{array}\right)$$

This is the inverse.

$$A^{-1} = \begin{pmatrix} 2/3 & 2/3 & -1 \\ -1/3 & -1/3 & 1 \\ 4/3 & 1/3 & -2 \end{pmatrix}$$

Finding the inverse of a matrix A is a lot of work but as we noted earlier, once we have it then solving linear systems $A\vec{x} = \vec{b}$ is easy.

Example The linear system

$$\begin{aligned} x + 3y + z &= \ \ 2 \\ 2x \qquad\quad - z &= 12 \\ x + 2y \qquad\ &= \ \ 4 \end{aligned}$$

is this matrix equation.

$$\begin{pmatrix} 1 & 3 & 1 \\ 2 & 0 & -1 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2 \\ 12 \\ 4 \end{pmatrix}$$

Solve it by multiplying both sides from the left by the inverse that we found earlier.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2/3 & 2/3 & -1 \\ -1/3 & -1/3 & 1 \\ 4/3 & 1/3 & -2 \end{pmatrix} \begin{pmatrix} 2 \\ 12 \\ 4 \end{pmatrix} = \begin{pmatrix} 16/3 \\ -2/3 \\ -4/3 \end{pmatrix}$$

We sometimes want to repeatedly solve systems with the same left side but different right sides. This system equals the one on the prior slide but for one number on the right.

$$\begin{aligned} x + 3y + z &= 1 \\ 2x \qquad - z &= 12 \\ x + 2y \qquad &= 4 \end{aligned}$$

The solution is this.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2/3 & 2/3 & -1 \\ -1/3 & -1/3 & 1 \\ 4/3 & 1/3 & -2 \end{pmatrix} \begin{pmatrix} 1 \\ 12 \\ 4 \end{pmatrix} = \begin{pmatrix} 14/3 \\ -1/3 \\ -8/3 \end{pmatrix}$$

The inverse of a 2×2 matrix

4.11 *Corollary* The inverse for a 2×2 matrix exists and equals

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

if and only if $ad - bc \neq 0$.

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Proof This computation is Exercise 21 . QED

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Example

$$\begin{pmatrix} 2 & 4 \\ -1 & 1 \end{pmatrix}^{-1} = \frac{1}{6}\begin{pmatrix} 1 & -4 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1/6 & -2/3 \\ 1/6 & 1/3 \end{pmatrix}$$

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Example

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The 3×3 formula is much more complicated. We will cover it in the next chapter.